Newton's differential equation $\frac{\dot{y}}{\dot{x}} = 1 - 3x + y + xx + xy$

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"But this will appear plainer by an Example or two."
Newton (1671)
After outlining his general method for finding
solutions of differential equations.

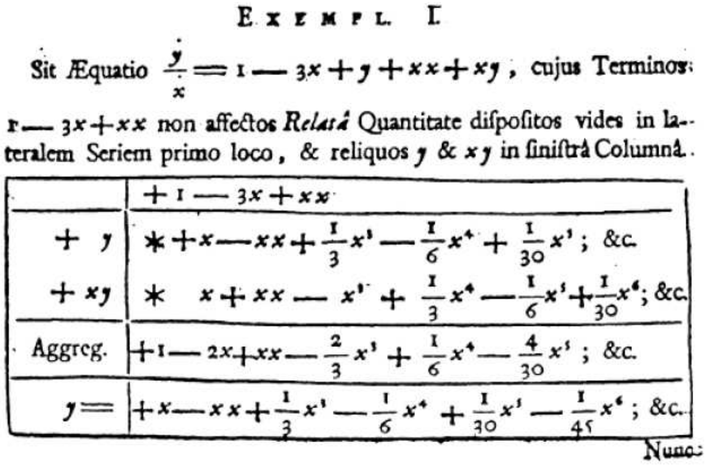
1 Introduction

Newton's book [5], *ANALYSIS Per Quantitatum, SERIES, FLUXIONES, AC DIFFERENTIAS: cum Enumeratione Linearum TERTII ORDINIS* consists of one dozen problems. The second problem

> *"PROB. II An Equation is being proposed, including the Fluxions of Quantities, to find the Relations of those Quantities to one another"*

is devoted to a general method of finding the solution of an initial-value problem for a scalar ordinary differential equation in terms of infinite series. The equation in the title of the present paper (see also Fig. 1) is the first significant example in the section on PROB. II.

Newton thought of Mathematical quantities as being generated by a continuous motion. He called such a flowing quantity a *fluent* (variable), and referred to its rate of change as the *fluxion* of fluent of the quantity and denoted it by a dot over the quantity. He denoted the change of *Relate Quantity* (dependent variable) with respect to the *Correlate Quantity* (independent variable) with the ratio of their fluxions:



EXEMPL. I.

Sit Æquatio $\frac{\dot{y}}{\dot{x}} = 1 - 3x + y + xx + xy$, cujus Terminos $1 - 3x + xx$ non affectos *Relatâ* Quantitate diſpoſitos vides in lateralem Seriem primo loco, & reliquos y & xy in ſiniſtrâ Columnâ.

	$+1 - 3x + xx$
$+y$	$* + x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$; &c.
$+xy$	$* \quad x + xx - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6$; &c.
Aggreg.	$+1 - 2x + xx - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5$; &c.
$y =$	$+x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$; &c.

Nunc

Figure 1: Original text of Newton's differential equation.

Quantities	x	Correlate Quantity
	y	Relate Quantity
	$\dot{x}$	Fluxion of x
	$\dot{y}$	Fluxion of y
Equation	$\frac{\dot{y}}{\dot{x}} = 1 - 3x + y + xx + xy$	

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Let us interpret Newton in our current calculus jargon. If we consider the relate quantity $y(t)$ and the correlate quantity $x(t)$ to be generated by continuous motions in time t then their fluxions $\dot{y}$ and $\dot{x}$ are

$$\dot{y} = \frac{dy}{dt}, \qquad \dot{x} = \frac{dx}{dt}$$

and the ratio of their fluxions becomes

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}.$$

Thus, Newton's proposed equation, "including the Fluxions of Quantities," can be written as

$$\frac{dy}{dx} = 1 - 3x + y + x^2 + xy$$

22 whose solution $y(x)$ will yield "the Relations of those Quantities to one another."

2 Newton's Solution

Newton obtained the solution of a differential equation satisfying a given initial condition in terms of infinite series. At each stage of his series solution, he inserted the series into his differential equation and integrated the resulting polynomial.

Now, we will paraphrase [3] Newton's steps and obtain several terms of his power series solution y(x) of his differential equation satisfying the initial condition $y(0) = 0$. Start with the first term

$$y = 0 + \cdots$$

and insert it into the differential equation to obtain

$$\frac{dy}{dx} = 1 + \cdots.$$

Now, integrate this with respect to x,

$$y = x + \cdots$$

to obtain the next term in the series. Inserting this series for y into the differential equation, yields

$$\frac{dy}{dx} = 1 - 2x + \cdots$$

integration of which gives

$$y = x - x^2 + \cdots.$$

The next iteration of this process gives

$$\frac{dy}{dx} = 1 - 2x + x^2 + \cdots$$

and

$$y = x - x^2 + \tfrac{1}{3}x^3 + \cdots.$$

Newton continues several more iterations and arrives at the solution

$$y = x - x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{6}x^4 + \tfrac{1}{30}x^5 - \tfrac{1}{45}x^6 + \cdots.$$

2.1 Newton's Demonstration

It is prudent to verify that a proposed solution of a differential equation indeed satisfies the differential equation. Here is how Newton demonstrates the validity of his solution:

DEMONSTRATION

56. And thus we have solved the Problem, but the demonstration is still behind. And in so great a variety of matters, that we may not derive it synthetically, and with too great perplexity, from its genuine foundations, it may be sufficient to point it out thus in short, by way of Analysis. That is, when any Equation is propos'd, after you have finish'd the work, you may try whether from the derived Equation you can return back to the Equation propos'd ... And thus from $\dot{y} = 1 - 3x + y + xx + xy$ *is derived* $y = x - x^2 + (1/3)x^3 - (1/6)x^4 + (1/30)x^5 - (1/45)x^6, \&c.$ *And thence by Prob. I.* $\dot{y} = 1 - 2x + x^2 - (2/3)x^3 + (1/6)x^4 - (2/15)x^5, \&c.$ *Which two values of* $\dot{y}$ *agree with each other, as appears by substituting* $x - xx + (1/3)x^3 - (1/6)x^4 + (1/30)x^5, \&c.$ *instead of* y *in the first value.*

3 Phaser Simulations

A series solution of an initial-value problem, in principle, should yield better approximations to the solution as more terms of the series are included. In Fig. 2, third through sixth-order series approximations of the solution of Newton's differential equation satisfying the initial condition $y(0) = 0$ are plotted.

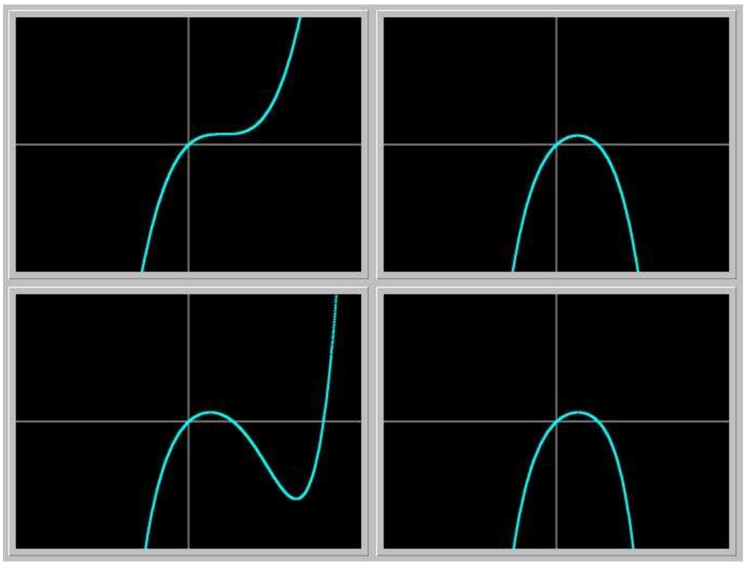

Figure 2: Third through sixth-order polynomial approximations of the Newton's series solution $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6 + \cdots$ are plotted.

A carefully computed actual solution of the differential equation satisfying the initial condition $y(0) = 0$ is plotted as the blue (lower) curve in Fig. 3. It was indicated above that one can expect better approximations as more terms of the series are included. However, this expectation holds only locally near the initial condition, but not globally. Indeed, the fourth-order approximation appears to resemble the actual solution more than the fifth-order approximation.

Newton also computed a series solution of his differential equation satisfying the initial condition $y(0) = 1$. A carefully computed graph of this solution is plotted in yellow (upper curve) in Fig. 3. More generally, Newton computed an infinity of solutions of his differential equation satisfying the initial condition $y(0) = a$ for any real number a. More information about these solutions are contained in the Suggested Explorations below.

At *http://www.phaser.com/modules/history/newton/index.html* an interactive version of this paper is available. With simple mouse clicks on Fig. 3 at this Phaser Web site [1], you can generate accurate solutions of Newton's differential equation satisfying any initial condition.

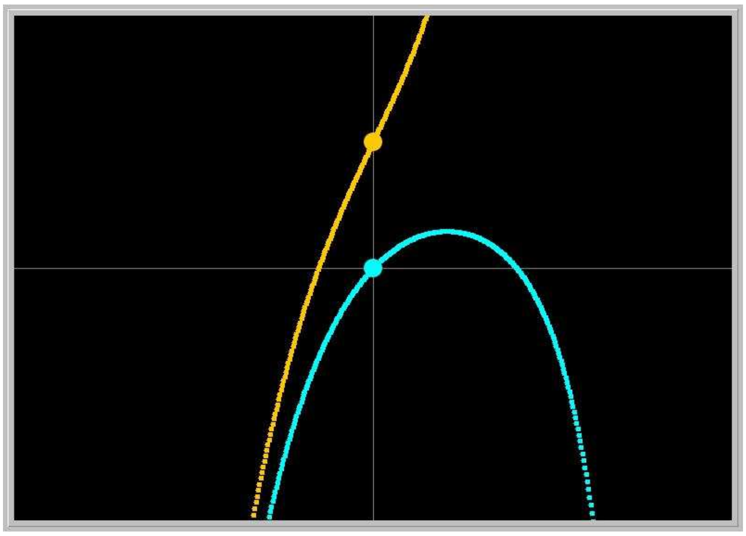

Figure 3: A carefully computed solution of Newton's differential equation $\dfrac{dy}{dx} = 1 - 3x + y + x^2 + xy$ satisfying the initial condition $y(0) = 0$ is plotted in blue (lower curve). The additional solution in yellow (upper curve) satisfies the initial condition $y(0) = 1$; Newton's series of this solution is given in the Suggested Explorations below.

4 Remarks: Newton, Leibniz, and Euler

Newton's differential equation is a scalar *linear* differential equation for which there exists a formula for the solutions. Indeed, using this formula, one obtains the following closed-form solution of Newton's differential equation satisfying the initial condition $y(0) = 0$:

$$y(x) = 4 - x + e^{(x+1)^2/2} \left(3\sqrt{2\pi}\left[\operatorname{erf}((x+1)/\sqrt{2}) - \operatorname{erf}(1/\sqrt{2})\right] - 4e^{-1/2}\right).$$

Notice, however, that the solution above involves the *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2/2} dt$$

which cannot be expressed in terms of elementary functions. Full details of the calculations leading to this solution is available at the Phaser Web site [1].

Like Newton, Leibniz also devoted a great deal of his attention to solving differential equations. His approach, however, was quite different from that of Newton's. Leibniz seeked mostly closed-from solutions in terms of known functions; in fact, he is often credited with the discovery of the method of separation of variables. "One of the earliest discoveries in the integral calculus was that the integral of a given function could only in very special cases be finitely expressed in terms of known functions. So it is also in the theory of differential equations. That any particular equation should be integrable in a finite form is to be regarded as a happy accident; in the general case the investigator has to fall back, as in the example just quoted, upon solutions expressed in infinite series whose coefficients are determined by recurrence formulae [4]." Indeed, Newton could "solve" any differential equation (see the Suggested explorations below) usign his power series method, including the ones that Leibniz could not integrate. It is interesting to speculate whether Newton suspected that his differential equation could not be integrated in terms of elementary functions.

Newton's power series method can generate approximate solutions of any desired accuracy; however, the series solution is valid only near a given inital condition. Another method of generating approximate solutions of differential equation is the method of Euler[2] which is commonly presented as the simplest algorithm in numerical analysis of differential equations. It is likely that Euler might have been trying to rectify the shortcoming of the locality of the power series method by devising a new approximation method capable of generating solutions away from the initial condition. Indeed, Euler writes [2]:

> *"... thus we can progress to values as distant from the initial values as we wish."*

Unlike Newton, Euler does not present a specific differential equation to demonstrate the effectiveness of his method. However, he does point out a new kind of difficulty with his method in the following Corollary:

> *Corollary 2. 652. Where smaller intervals are taken, through which the values of x progress iteratively, so much the more accurate values are obtained one at a time. However the errors committed one at a time, even if they may be very small, accumulate because of the multitude.*

5 Suggested Explorations

1. Newton solved his equation for the initial value $y(0) = 1$ as well. His answer, in this case, is $y = 1 + 2x + x^3 + \frac{1}{4}x^4 + \frac{1}{4}x^5 + \cdots$.

   Demonstrate the validity of Newton's solution a la Newton. This solution is plotted in yellow (upper curve) in Fig. 3 above.

2. It is very interesting to observe that Newton calculates up to sixth-order (even) terms for the blue solution while he stops at the fifth-order (odd) terms for the yellow solution. Series solutions should become more accurate with additional terms; this may be true locally but not necessarily globally. Why do you think Newton stopped at the fifth-order terms for the yellow solution while continued to the sixth-order terms for the blue solution?

3. Visit *http://www.phaser.com/modules/history/newton/index.html* and load Fig. 3 into your local copy of Phaser by simply clicking on the picture. Now, click the left mouse button at several locations along the vertical axis to mark additional initial conditions. Press the Go button of Phaser to see the additional solutions.

4. Newton also computed the solution of his differential equation for the initial condition $y(0) = a$:

   > *"I said before, that these Solutions may be performed by an infinite variety of ways. This may be done if you assume at pleasure not only the initial quantity of the upper series, but any other given quantity for the first Term of the Quote, and then you may proceed as before. ...Or if you make use of any Symbol, say a, to represent the first Term indefinitely, by the same method of Operation (which I shall here set down,)* $y = a + x + ax - xx + axx + (1/3)x^3 + (2/3)ax^3 + \cdots$ *which being found, you may substitute 1, 2, 0, (1/2), or any other number, and thereby obtain the Relation between x and y an infinite variety of ways."*

   Verify his answer.

5. Find the solution satisfying the general initial condition $y(x_0) = y_0$. Hint: Find the power series expansion in powers of $(x - x_0)$.

6. Newton also studied differential equations whose right-hand-sides are more complicated than polynomials in x and y. In this case, he first expanded the differential equation into a power series and proceeded as before. Here is such an example.

   > *" 32. And after the same manner the Equation* $\dot{y}/\dot{x} = 3y - 2x + x/y - 2y/(xx)$ *being proposed; if, by reason of the Terms* x/y *and* $2y/(xx)$, *I write* $1 - y$ *for y,* $1 - x$ *for x, there will arise* $\dot{y}/\dot{x} = 1 - 3y + 2x + (1 - x)/(1 - y) + (2y - 2)/(1 - 2x + x^2$. *But the Term* $(1 - x)/(1 - y)$ *by infinite Division gives*

$1-x+y-xy+y^2-xy^2+y^3-xy^3$, &c. *and the Term* $(2y-2)/(1-2x+xx)$ *by a like Division gives* $2y-2+4xy-4x+6x^2y-6x^2+8x^3y-8x^3+10x^4y-10x^4$, &c. *Therefore* $\dot{y}/\dot{x} = -3x+3xy+y^2-xy^2+y^3-xy^3$, &c. $+ 6x^2y-6x^2+8x^3y-8x^3+10x^4y-10x^4$, &c. "

Perform the "infinite Divisions" and verify Newton's calculations.

References

[1] PHASER [2007]. *Phaser : A Universal Simulator for Dynamical Systems, Based on Java Technology*, at http://www.phaser.com.

[2] EULERI, L. [1824]. *Caput VII. de integratione Aequationum Differentialium per Approximationem* in *Calculi Integralis, Liber Prior. Sectio Secunda, de Integratione Aequationum Differentialium.* Impensis Academiae Imperialis Scientiarum, Petropoli. English translation of this article is available in the article by D. Jardine in this volume.

[3] HAIRER, E., NORSETT, S., and WANNER, G. [1987]. *Solving Ordinary Differential Equations.* Springer-Verlag, New York.

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[5] NEWTON, I. [1736]. *The Method of Fluxions and Infinite Series; with its Applications to the Geometry of Curve-lines* by the Inventor Sir Isaac Newton, $K^{t.}$, Late President of the Royal Society. Translated from the Author's Latin Original not yet made publick. To which is subjoin'd, A perpetual Comment upon the whole Work, Consisting of Annotations, Illustrations, and Supplements, In Order to make this Treatise A Compleat Institution for the use of Learners. By John Colson. London: Printed by Henry Woodfall; And Sold by John Nourse, at the Lamb without Temple-Bar. M.DCCXXVI.

[6] O'CONNOR, J.J. and ROBERTSON, E.F. *Sir Isaac Newton.* http://www-gap.dcs.st-and.ac.uk/ history/Mathematicians/Newton.html