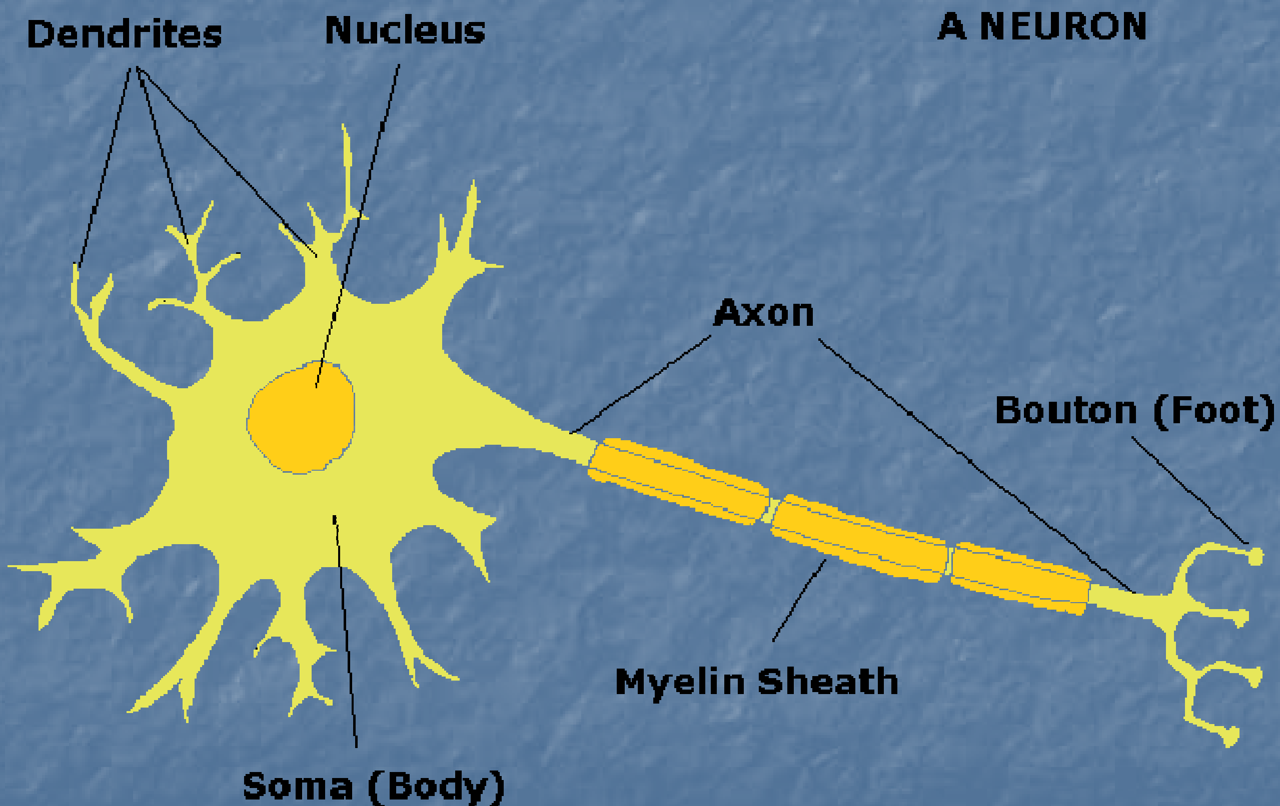
Hodgkin-Huxley Model and FitzHugh-Nagumo Model

Nervous System

- Signals are propagated from nerve cell to nerve cell (*neuron*) via electro-chemical mechanisms
- ~100 billion neurons in a person
- Hodgkin and Huxley experimented on squids and discovered how the signal is produced within the neuron
- H.-H. model was published in *Jour. of Physiology* (1952)
- H.-H. were awarded 1963 Nobel Prize
- FitzHugh-Nagumo model is a simplification

Neuron



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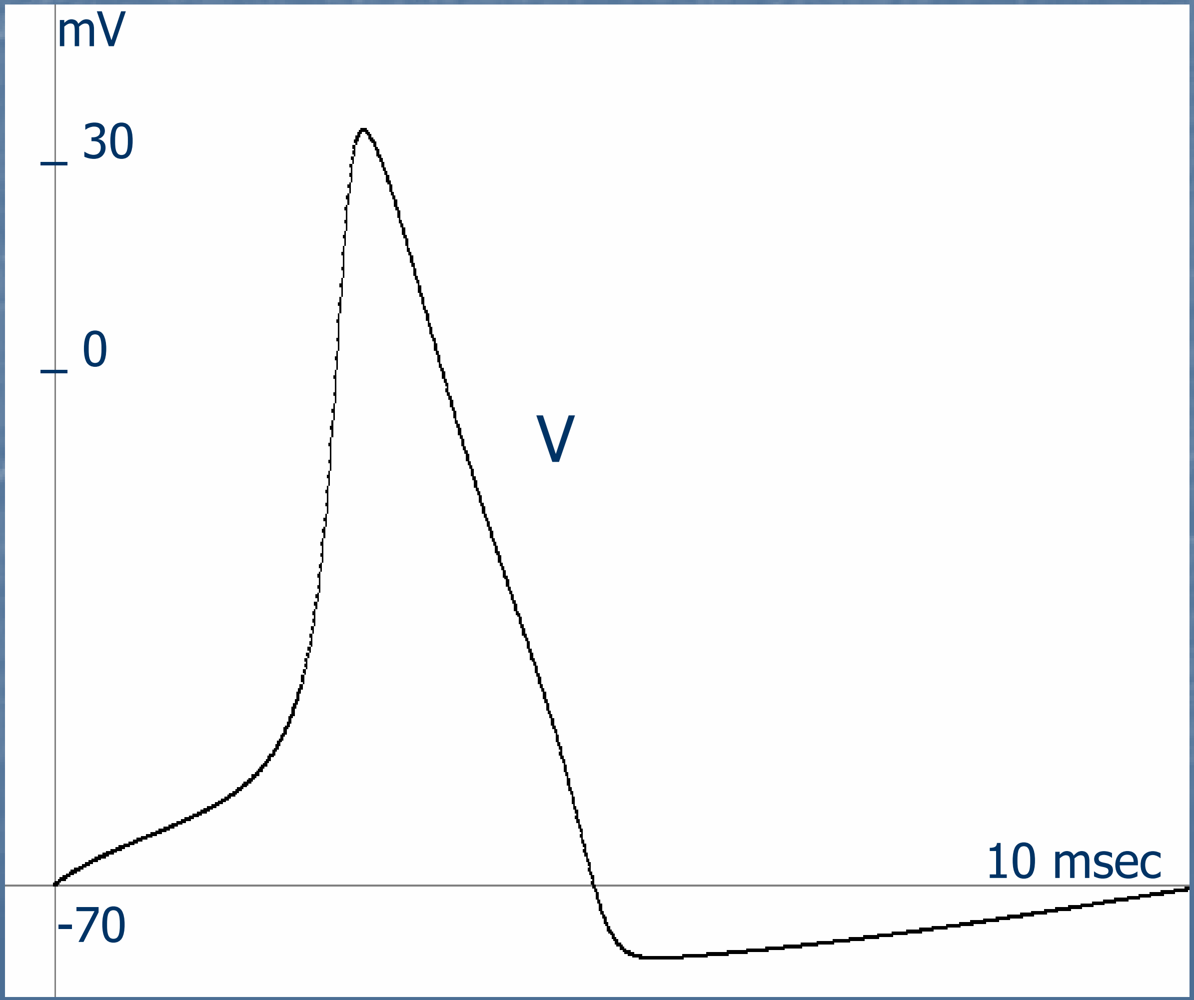
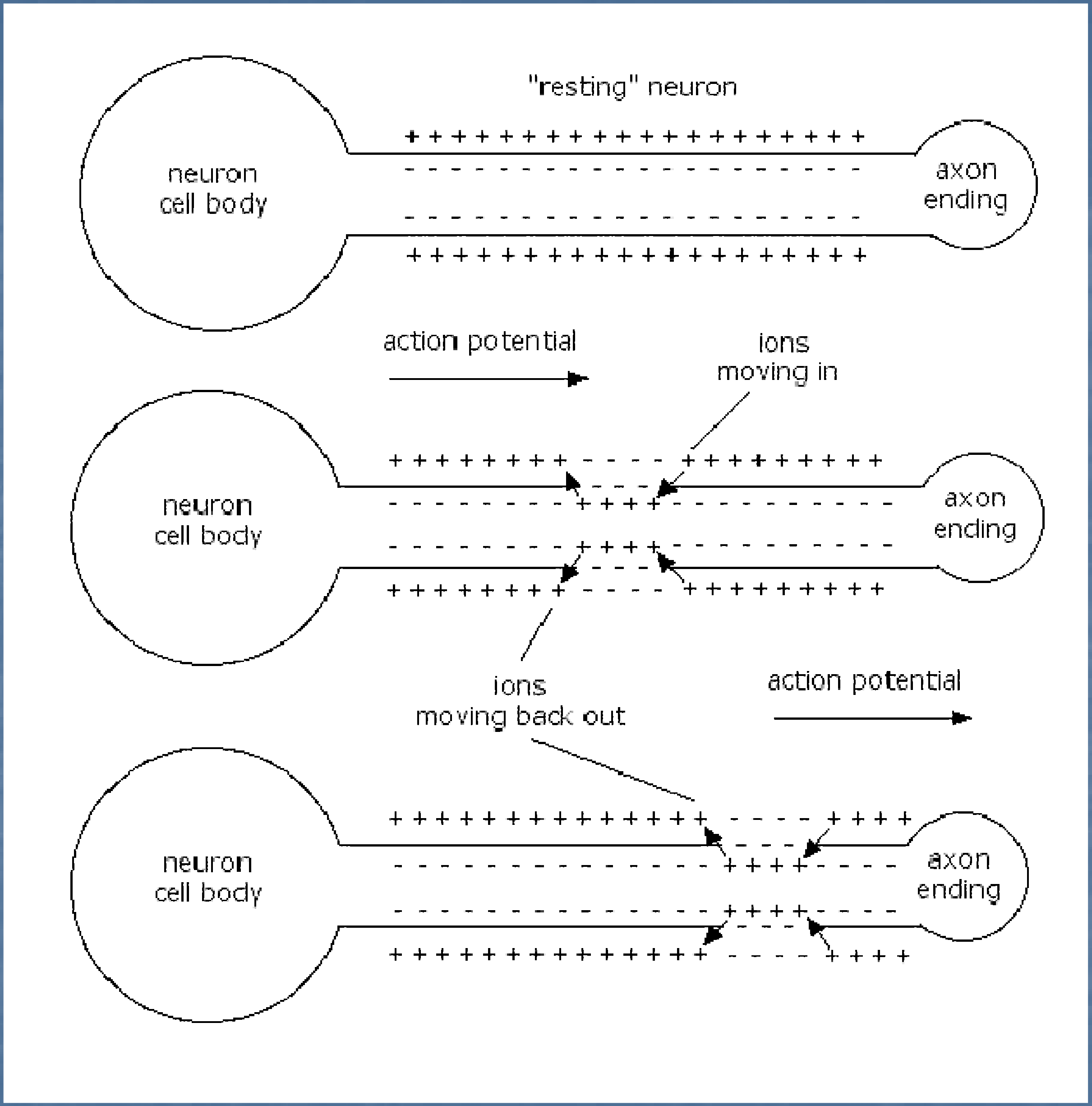
Action Potential



Axon membrane potential difference

$$V = V_i - V_e$$

When the axon is excited, V spikes because sodium Na+ and potassium K+ ions flow through the membrane.



Nernst Potential

V_{Na} , V_K and V_r

Ion flow due to electrical signal

Traveling wave

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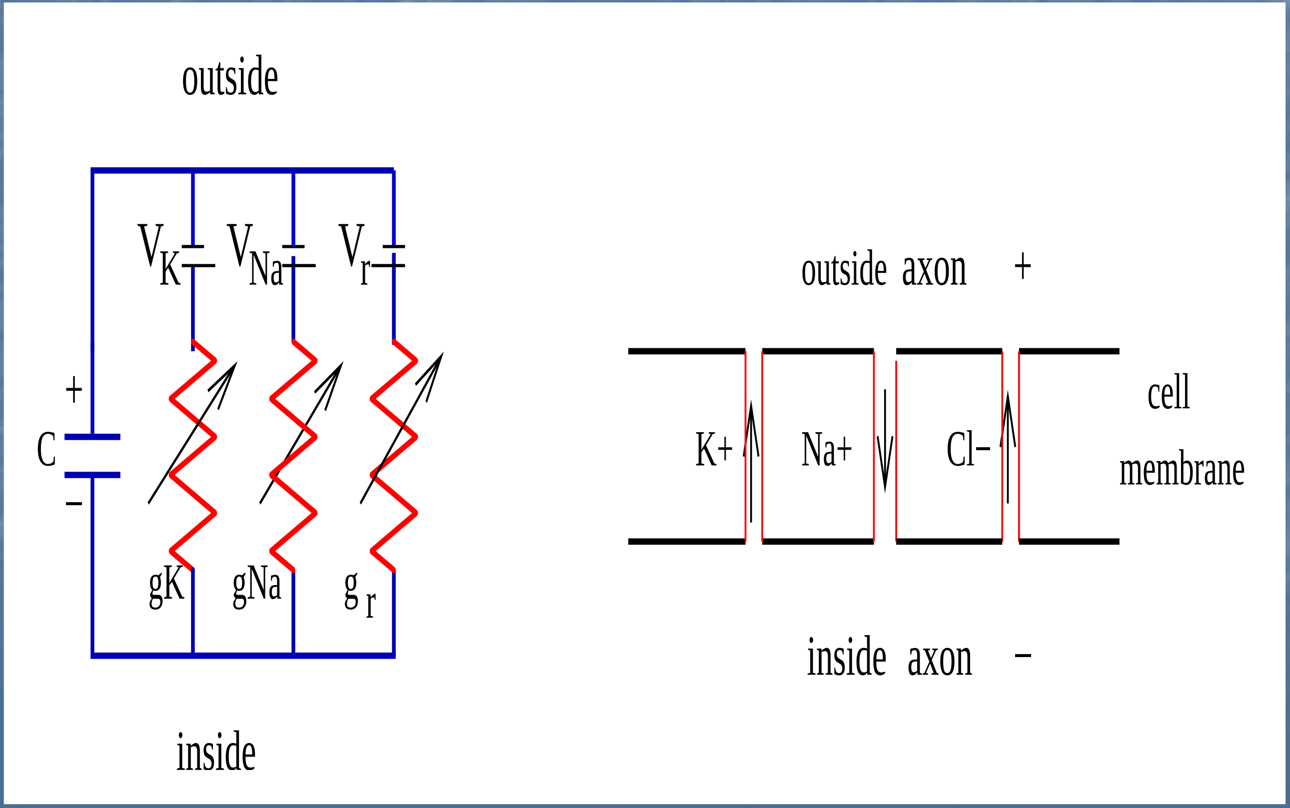
Circuit Model for Axon Membrane

Since the membrane separates charge, it is modeled as a capacitor with capacitance C. Ion channels are resistors.

$$1/R = g = \text{conductance}$$



$i_C = C\,dV/dt$

$i_{Na} = g_{Na}(V - V_{Na})$

$i_K = g_K(V - V_K)$

$i_r = g_r(V - V_r)$

Circuit Equations

Since the sum of the currents is 0, it follows that

$$C\frac{dV}{dt} = -g_{Na}(V - V_{Na}) - g_K(V - V_K) - g_r(V - V_r) + I_{ap}$$

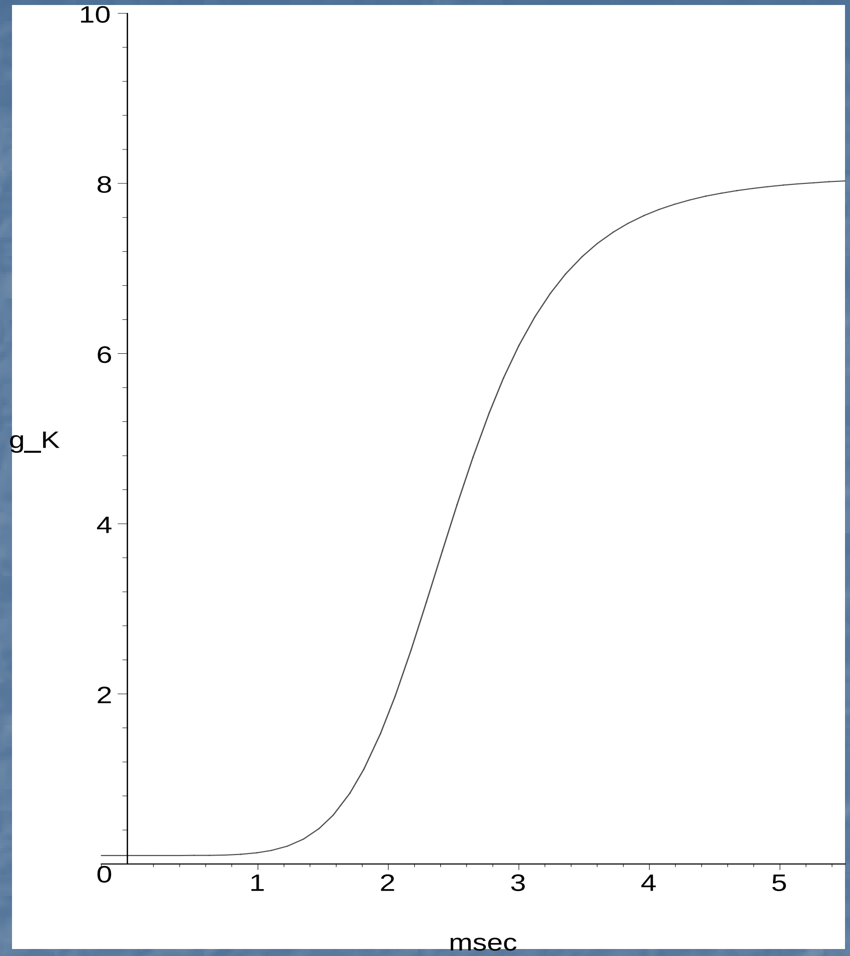
where I_{ap} is applied current. If ion conductances are constants then group constants to obtain 1st order, linear eq

$$C\frac{dV}{dt} = -g(V - V^*) + I_{ap}$$

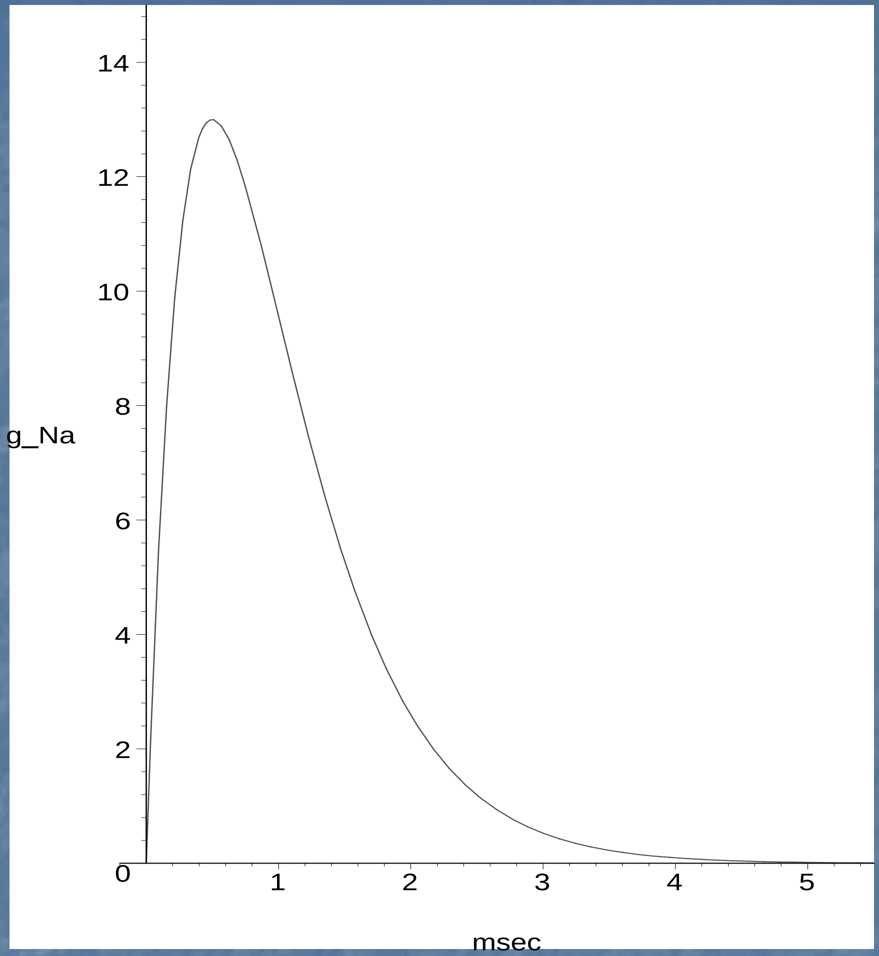
Solving gives

$$V(t) \to V^* + I_{ap}/g$$

Variable Conductance



g



Experiments showed that g_{Na} and g_K varied with time and V. After stimulus, Na responds much more rapidly than K .

Hodgkin-Huxley System

Four state variables are used:

$v(t)=V(t)-V_{eq}$ is membrane potential,

$m(t)$ is Na activation,

$n(t)$ is K activation and

$h(t)$ is Na inactivation.

In terms of these variables $g_K=\underline{g}_K n^4$ and $g_{Na}=\underline{g}_{Na} m^3 h$. The resting potential $V_{eq}\approx-70$mV. Voltage clamp experiments determined g_K and n as functions of t and hence the parameter dependences on v in the differential eq. for $n(t)$. Likewise for $m(t)$ and $h(t)$.

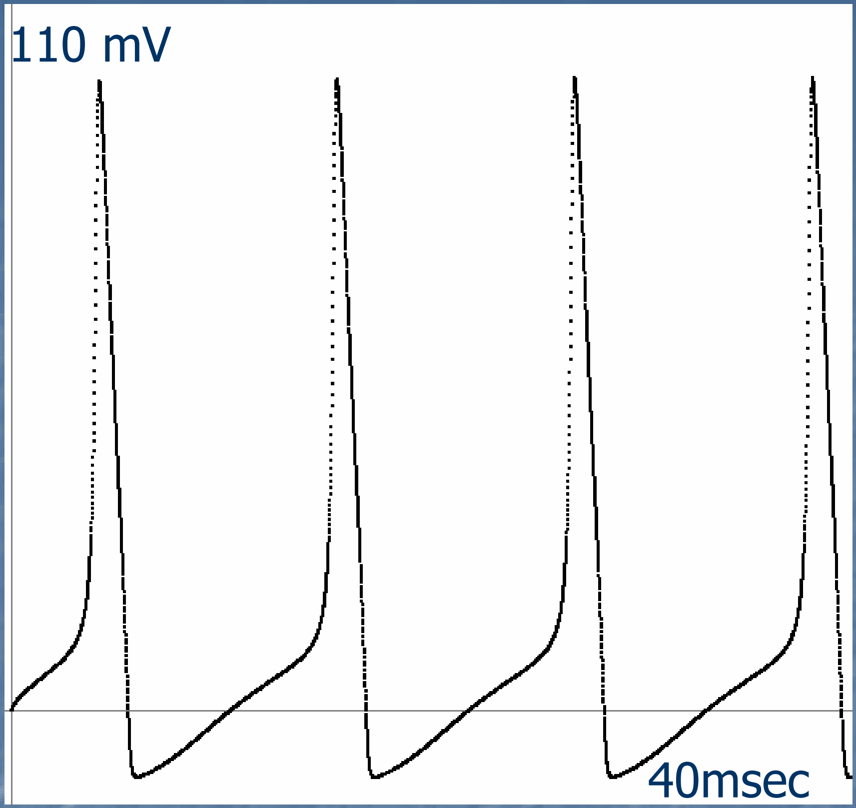
Hodgkin-Huxley System

$$C\frac{dv}{dt} = -\underline{g}_{Na} m^3 h (v - V_{Na}) - \underline{g}_K n^4 (v - V_K) - g_r (v - V_r) + I_{ap}$$

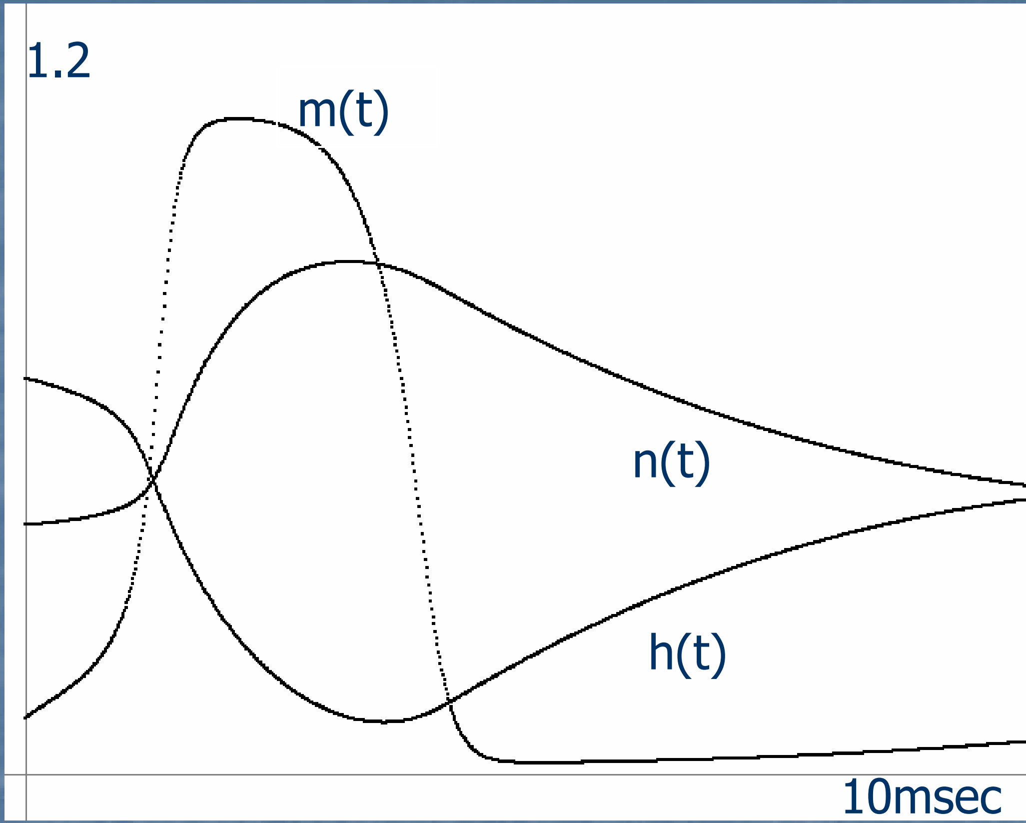
$$\frac{dm}{dt} = \alpha_m(v)(1 - m) - \beta_m(v)m$$

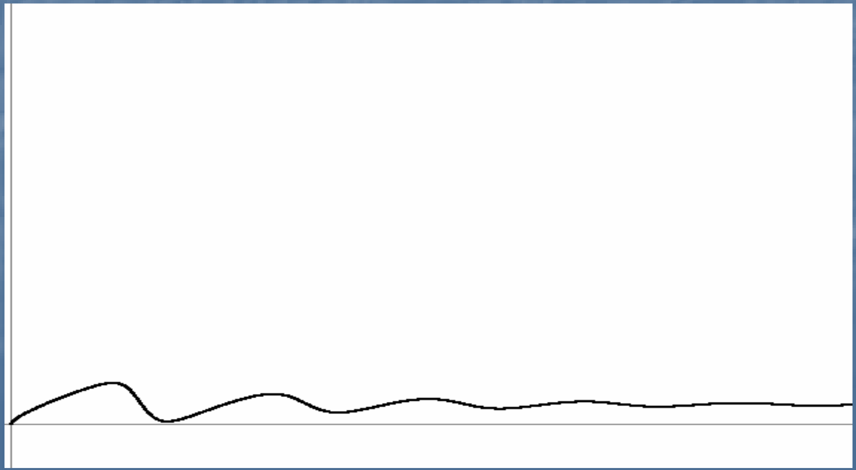
$$\frac{dn}{dt} = \alpha_n(v)(1 - n) - \beta_n(v)n$$

$$\frac{dh}{dt} = \alpha_h(v)(1 - h) - \beta_h(v)h$$



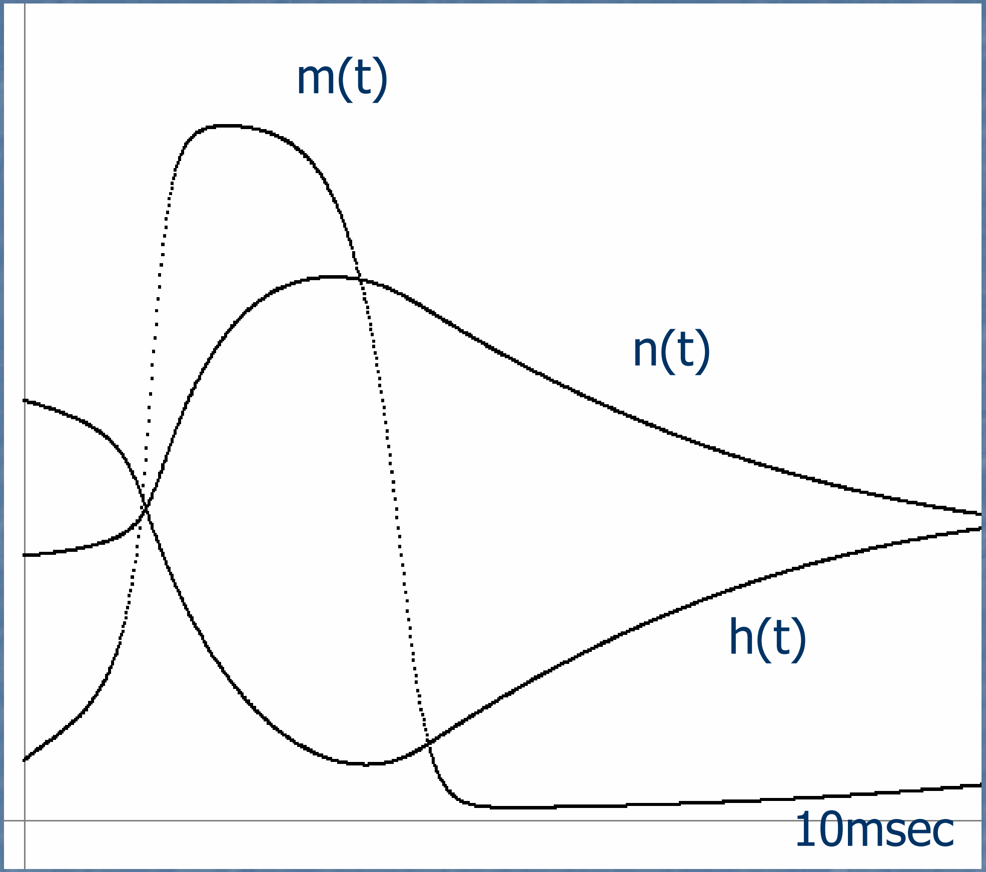
I_{ap} =8, v(t)





I_{ap}=7, v(t)

Fast-Slow Dynamics



$\rho_m(v)\, dm/dt = m_\infty(v) - m.$

$\rho_m(v)$ is much smaller than

$\rho_n(v)$ and $\rho_h(v)$. An increase in v results in an increase in $m_\infty(v)$ and a large dm/dt. Hence Na activates more rapidly than K in response to a change in v.

v, m are on a fast time scale and n, h are slow.

FitzHugh-Nagumo System

$$\varepsilon\frac{dv}{dt}=f(v)-w+I \quad \text{and} \quad \frac{dw}{dt}=v-0.5w$$

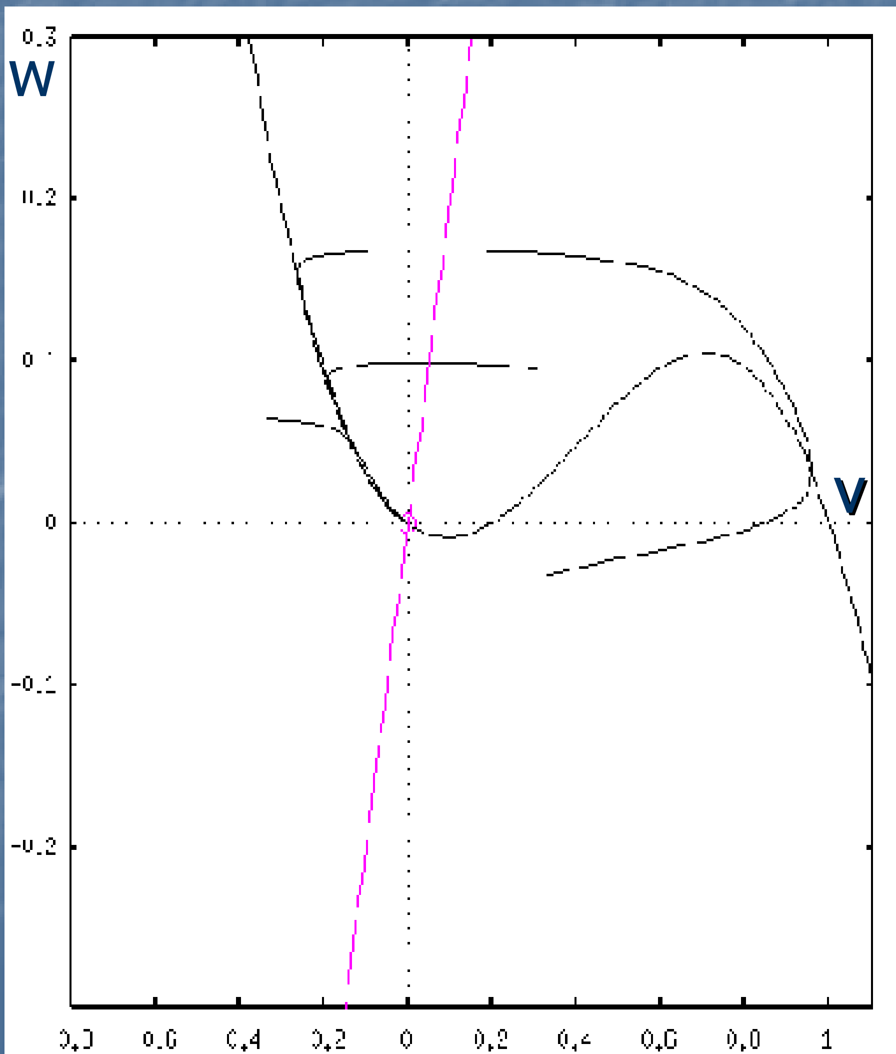
I represents applied current, ε is small and f(v) is a cubic nonlinearity. Observe that in the (v,w) phase plane

$$\frac{dw}{dv}=\frac{\varepsilon(v-0.5w)}{f(v)-w+I}$$

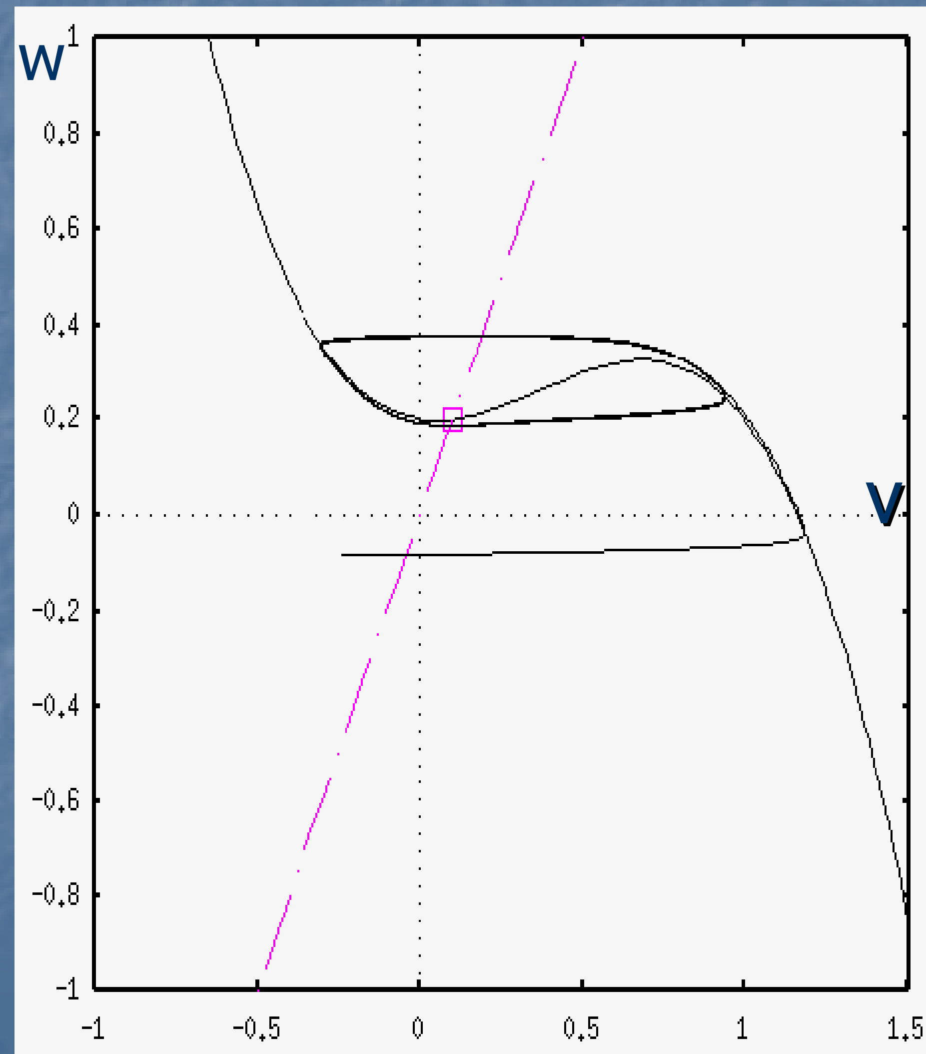
which is small unless the solution is near f(v)-w+*I*=0. Thus the *slow manifold* is the cubic w=f(v)+*I* which is the *nullcline* of the fast variable v. And w is the slow variable with *nullcline* w=2v.

Take f(v)=v(1-v)(v-a) .

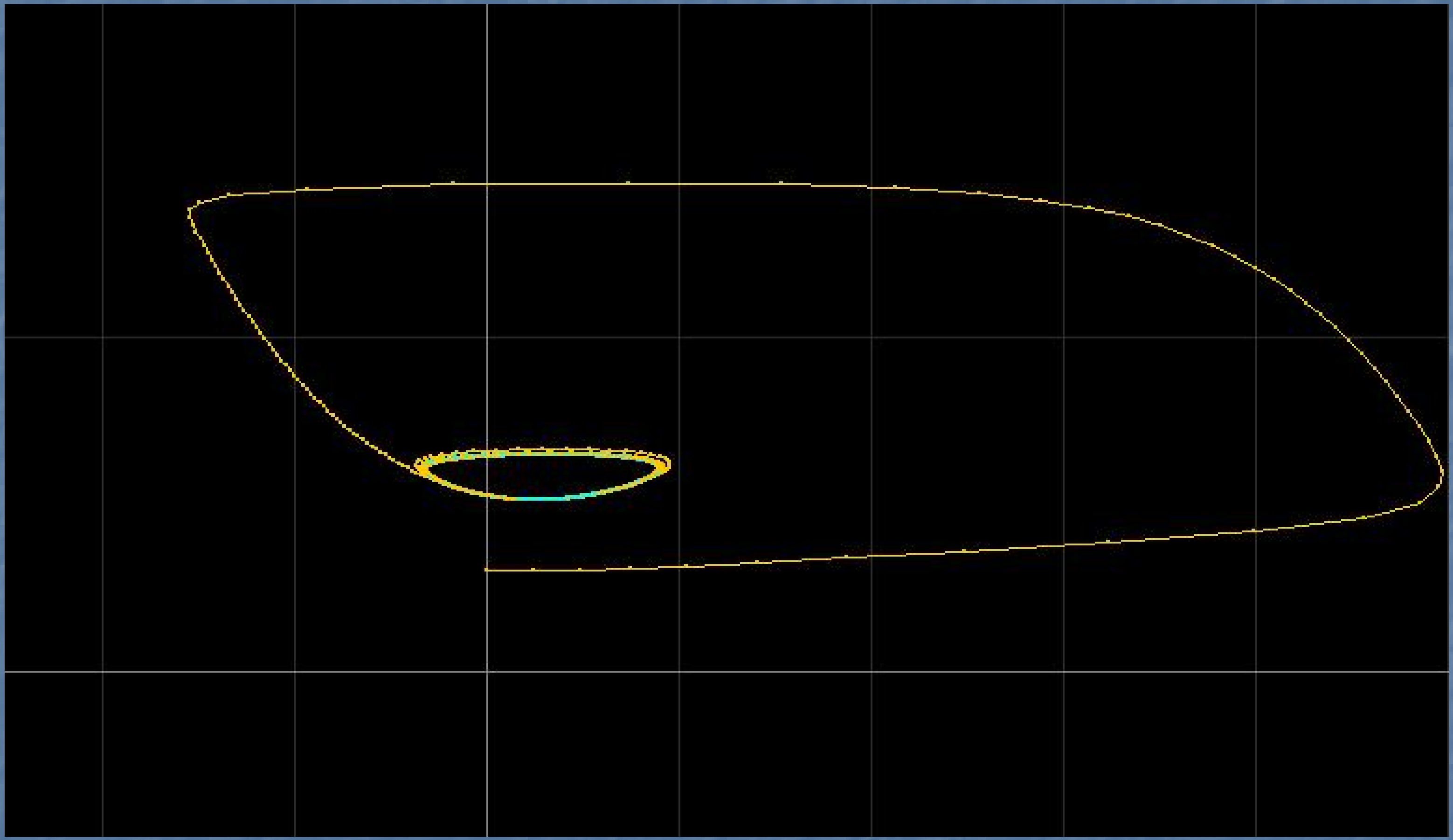
Stable rest state I=0



Stable oscillation I=0.2



FitzHugh-Nagumo Orbits



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