Phaser

A Universal Simulator For Dynamical Systems

Built on Java Technology

Getting Started with Phaser

Version 3

Contents

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Welcome to Phaser

Thank you for considering Phaser as an addition to your Desktop.

Phaser provides a powerful, yet inviting, computing environment specifically crafted for the graphical and numerical simulations of differential and difference equations, from linear to chaotic.

Phaser has been designed to meet the needs of a wide variety of users whether they be general chaos enthusiasts, students, instructors, or researchers. Written in pure Java, Phaser can be used on multiple platforms, or to share projects freely across platforms.

Chaos enthusiasts will appreciate Phaser's unprecedented ease of use, with graphics purposely set up to enhance the pleasures of learning about chaos and fractals. They will enjoy lively demonstrations revealing the dynamics of landmark equations.

Students will exploit Phaser's potential to manipulate standard textbook examples and exercises in order to achieve a better understanding of differential equations and their applications. They will find Phaser's step-by-step tutorials particularly useful. Instructors of undergraduate and graduate classes will be pleased to learn that Phaser provides powerful media specifically designed for effective classroom or laboratory presentations. The Phaser Project environment and the Gallery can be used to store, retrieve and animate a variety of demonstration elements. Instructors will also appreciate that Phaser includes a comprehensive, expandable, library of equations and a reference guide to computational dynamical systems.

Researchers in fields as diverse as biology, chemistry, economics, engineering, mathematics, and physics will find Phaser an invaluable tool in their quest to visualize quantitative models quickly and accurately. An equation editor allows new equations to be entered without programming. State-of-the-art numerical algorithms provide research-grade results with completely configurable error controls. Phaser's fingertip controls allow for rapid exploration of global dynamics and bifurcations, as well as for the sharing of research findings across the Internet.

In the succeeding pages you will find a brief overview of Phaser and some guidance to help you quickly experience Phaser's user-friendly computing environment. Afterwords, with the assistance of the Tutorials, you are invited to explore some of the more prominent capabilities of Phaser.

Customer Support

Phaser support team provides several forms of technical assistance:

- Visit our main Web site www.phaser.com and consult FAQs (Frequently Asked Questions) and troubleshooting information that provides answers to common problems.
- Read tips from the menu item **Help -> Tips**.
- Consult PhaserHelp from the menu item **Help -> Help Contents**. In PhaserHelp, browse Index or Search for key words.
- E-mail your Phaser questions to support@phaser.com. In your correspondence, please provide system information which can be gathered from the menu item **Help -> System Information**.
- Register at www.phaser.com to receive occasional Phaser news.

Licenses

License Agreement

A copy of Phaser purchased either on a CD or downloaded from our Web site www.phaser.com is for personal use on a single computer. To install Phaser on your computer, you are required to agree to the terms and conditions of the *Phaser License Agreement*.

Site License

If Phaser will be used on multiple computers or by multiple users, a site license must be obtained. To accomodate varying needs, several levels of site licencing are available. Please visit our Web site www.phaser.com for futher details of site licensing Phaser, and the terms and conditions of the *Phaser Site License Agreement*. If you require further information or terms customized for your needs, please contact us at info@phaser.com.

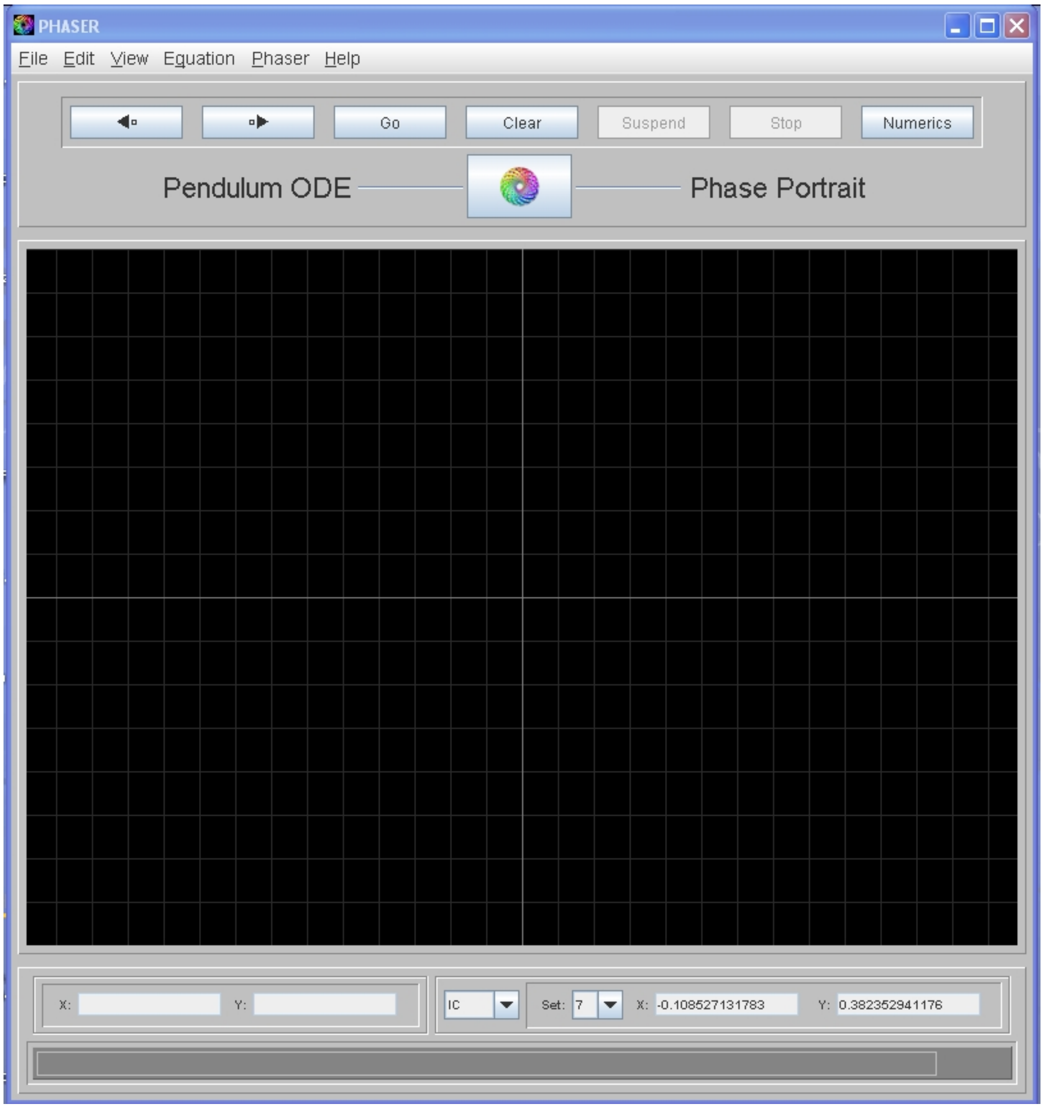
Sun Microsystems Binary Code Agreement

As part of the Phaser installation setup, a copy of Java Runtime Environment (JRE) may be installed on your computer, if an appropriate JRE is not already present. In this case, you must also agree to read and abide by the terms and conditions of the *Binary Code License Agreement for the JAVA SE RUNTIME ENVIRONMENT (JRE) VERSION 6* of Sun Microsystems, Inc.; see, for example, http://java.sun.com/javase/6/jre-6u1-license.txt.

Additional information on Sun Microsystems Licensing and Terms of Use details can be found at http://www.sun.com/termsofuse.html/

Startup

Main: In this section we describe the startup screen layout of Phaser and establish a common terminology for its graphical user interface. When the loading of Phaser is completed, you should see the *PHASER Main Application window* as displayed below.



Main Application Window: PHASER

The Main Application window consists of several panels with the following components and functions:

Title Bar

This is the top bar containing the title *PHASER* of the main application window.

Menu Bar

Below the title bar is the menu bar containing pull-down menus `File`, `Edit`, `View`, `Equation`, `Phaser`, and `Help`.

Button Bar

This is the row of buttons labeled `<<`, `>>`, `Go`, `Clear`, `Suspend`, `Stop`, `Numerics.`

Banner

This is the area which contains the current equation name, `Pendulum ODE,` and the current view name, `Phase Portrait.` As the current equation or the current view is changed the banner is updated. The colorful Phaser icon button in the center is used to access the Phaser `Gallery`.

View

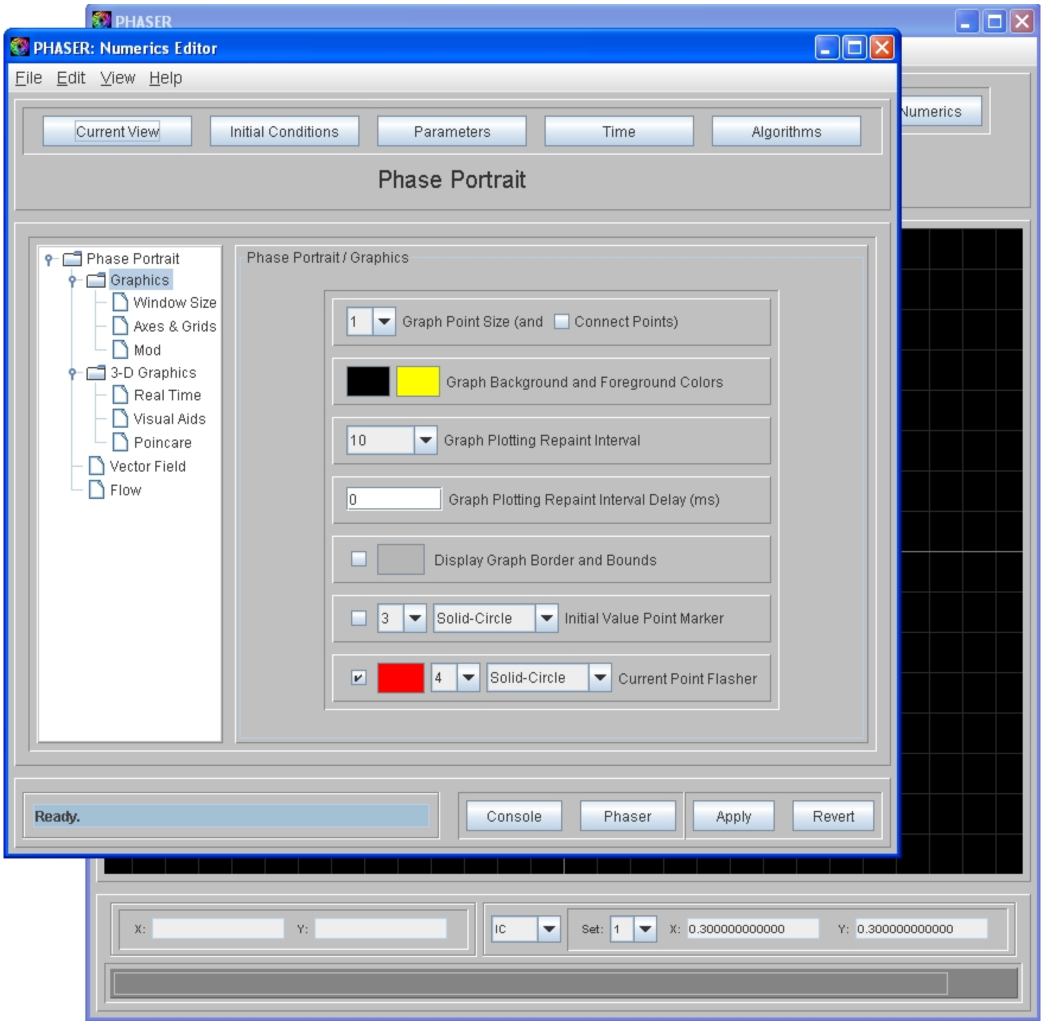
This is the large, black graphical display area containing axes and grids. The current view is `Phase Portrait`; the view can be changed from the `View` pull-down menu on the menu bar, or they can be cycled with the `<<` and `>>` buttons on the Button Bar.

Tool Bar

This is the panel immediately below the Phase Portrait view and it serves a dual purpose. The left portion is for status of mouse cursor location in the graphical view. The central choice button `IC` is a mouse action selector for using the mouse to set Initial Conditions, Zoom, Pan, 3-D rotations, or Flow. The right portion of the Tool Bar is dedicated for configuring the current mouse function.

Progress Bar

This is the last panel of the Main Application window which displays a progress bar and a percentage meter to monitor the status of the computations after the `Go` button is clicked.



Co-primary Window: PHASER: Numerics Editor

Numerics Editor: Now, if you press the `Numerics` button on the Button Bar of the main application window, a new *co-primary* window titled *PHASER: Numerics Editor* is spawned. This co-primary window is the "bridge" or command center of Phaser. The Numerics Editor window has the following panels:

Title Bar

This is the top bar containing the title *PHASER: Numerics Editor* of the co-primary window. You can click on it to bring the window to the foreground if it is partially hidden.

Menu Bar

This is the bar containing the pull-down menus `File`, `Edit`, `View`, and `Help`.

Button Bar

This bar contains the buttons `Current View`, `Initial Conditions`, `Parameters`, `Time`, and `Algorithms`. Each one of these buttons brings up a sub-editor which is displayed in the large center panel.

Banner

It displays the name of the selected button from the button bar. In the case `Current View` is selected, it displays the name of the current view (e.g. Phase Portrait).

Center Panel

The large panel in the center of the Numerics Editor window contains the appropriate sub-editor requested by one of the buttons on the button bar. Most of these sub-editors contain two main components. The first component on the left is a *tree* whose nodes contain various choices grouped together in a cohesive manner. The choices for each node are listed in the right portion of the central panel.

Status Field

This is the elongated box in the bottom left corner which contains the word ``Ready." This field provides feedback on the actions requested.

Action Buttons

These are the four buttons, `Console`, `Phaser`, `Apply`, and `Revert`, in the lower-right corner.

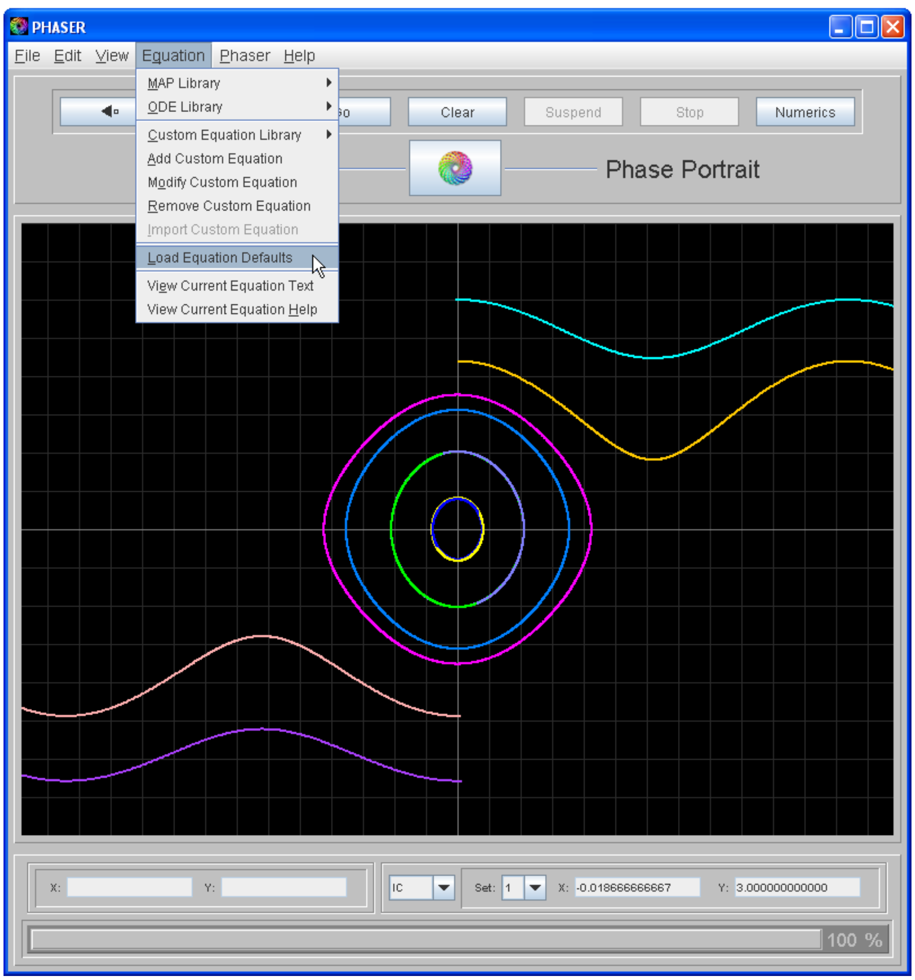
Browsing

Phaser is equipped with extensive libraries of equations of historical, scientific, or aesthetic appeal. Each equation comes with default settings that result in an informative picture. Here is how you can view these default pictures and get a first impression of some of the basic capabilities of Phaser.

Pendulum ODE

When Phaser starts, the default equation is Pendulum ODE. To see a partial phase portrait of the classical planar pendulum without friction, select the menu item

Equation → Load Equation Defaults



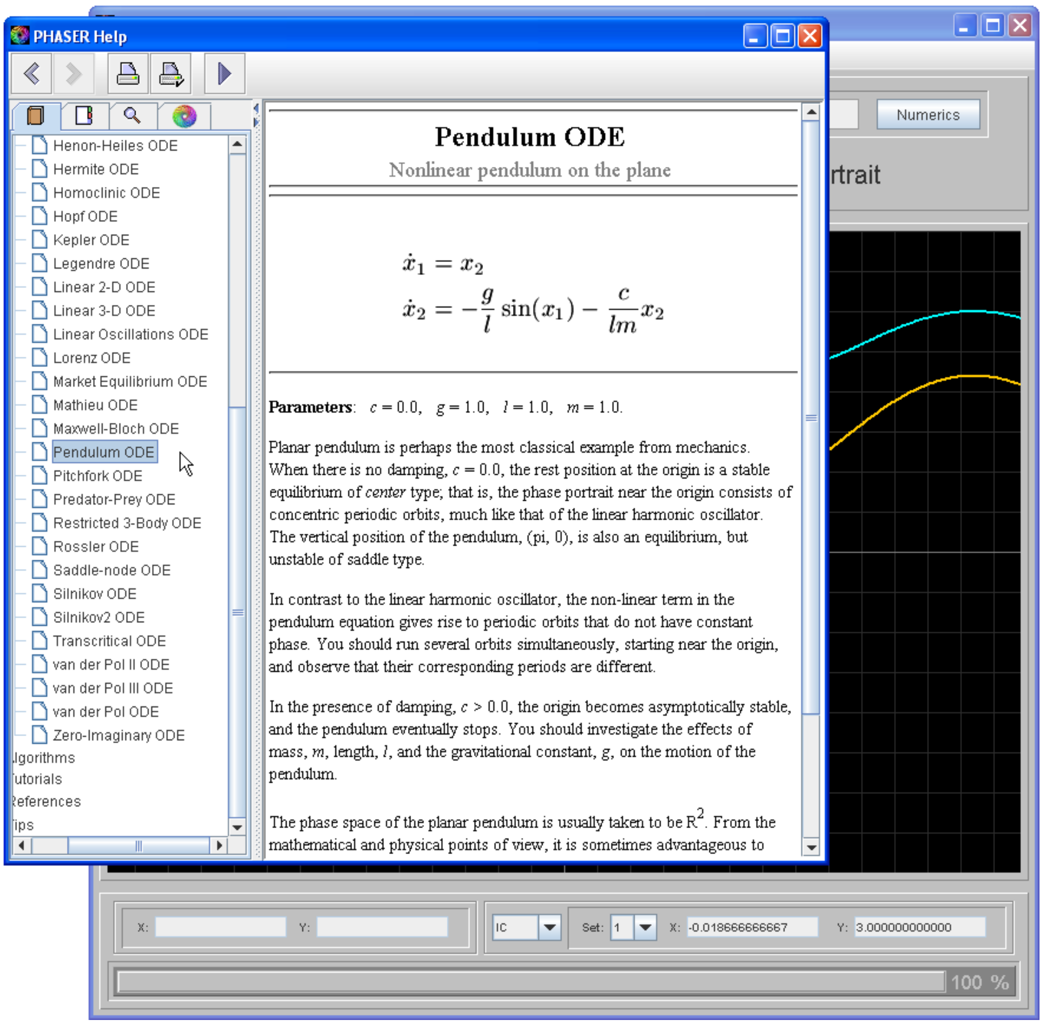
Phase portrait of Pendulum ODE

For additional solutions, click the left mouse button at desired points on the black region and press `Go`.

PhaserHelp

You can now view the page of PhaserHelp containing the mathematical equations, and thumbnail sketches of dynamical and physical highlights of the Pendulum ODE: select the menu item

Equation → View Current Equation Help



PhaserHelp entry for Pendulum ODE

PhaserHelp comprises an extensive online manual. In addition to mathematical, computational, and experimental information about each Library Equation, it contains detailed specifications of all Phaser commands, implemetation details of algorithms, Tutorials, and a collection of Tips for common and uncommon tasks. PhaserHelp can be launched by selecting the menu item

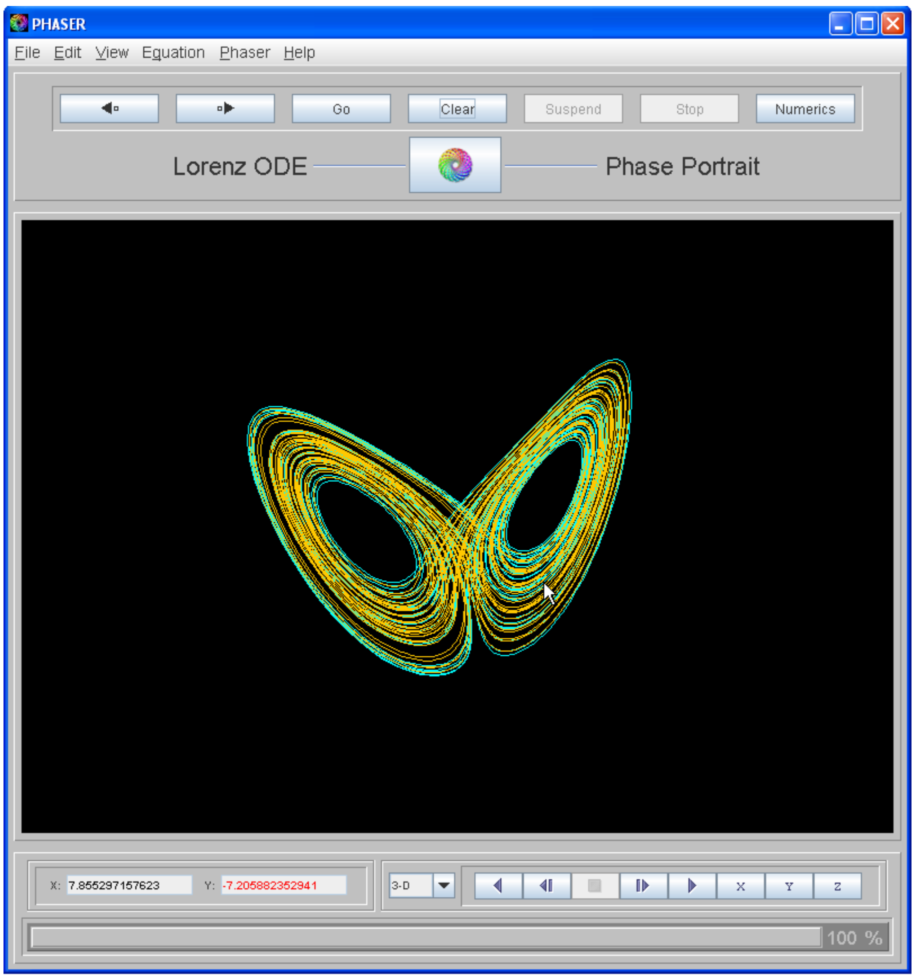
Help → Help Contents

and the desired information can easily be located using Table of Contents, Index, or searchable text.

Lorenz ODE

To load the famous Lorenz equations from the Phaser ODE Library and compute a pair of non-periodic solutions (one in blue and the other in yellow) in 3-D, select the menu items:

Equation → ODE Library → Lorenz ODE
Equation → Load Equation Defaults



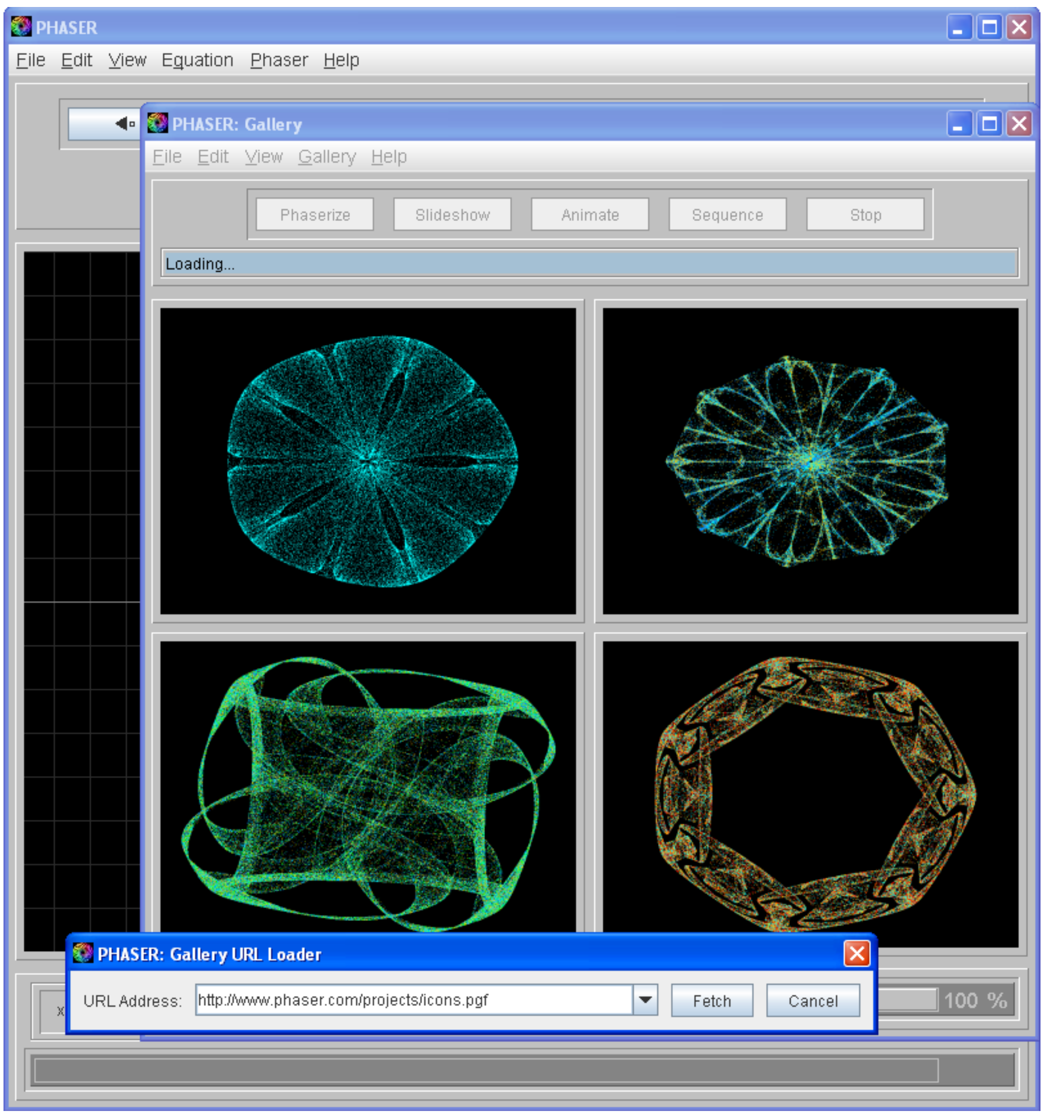
Two nonperiodic solutions of the Lorenz Equations

The mouse can be configured for several different functions in Phaser. For instance, in the Pendulum ODE simulation, it was set to select new initial conditions (IC). In the Equation Defaults of Lorenz ODE, the mouse is configured to rotate the image in three dimensions. This is indicated as 3-D in the mouse slector widget in the center of the tool bar. Now, grab the image with your mouse and rotate it while holding down the left mouse button. The radio buttons on the tool bar can be used for more precise animations of the image.

Symmetric Icons MAP

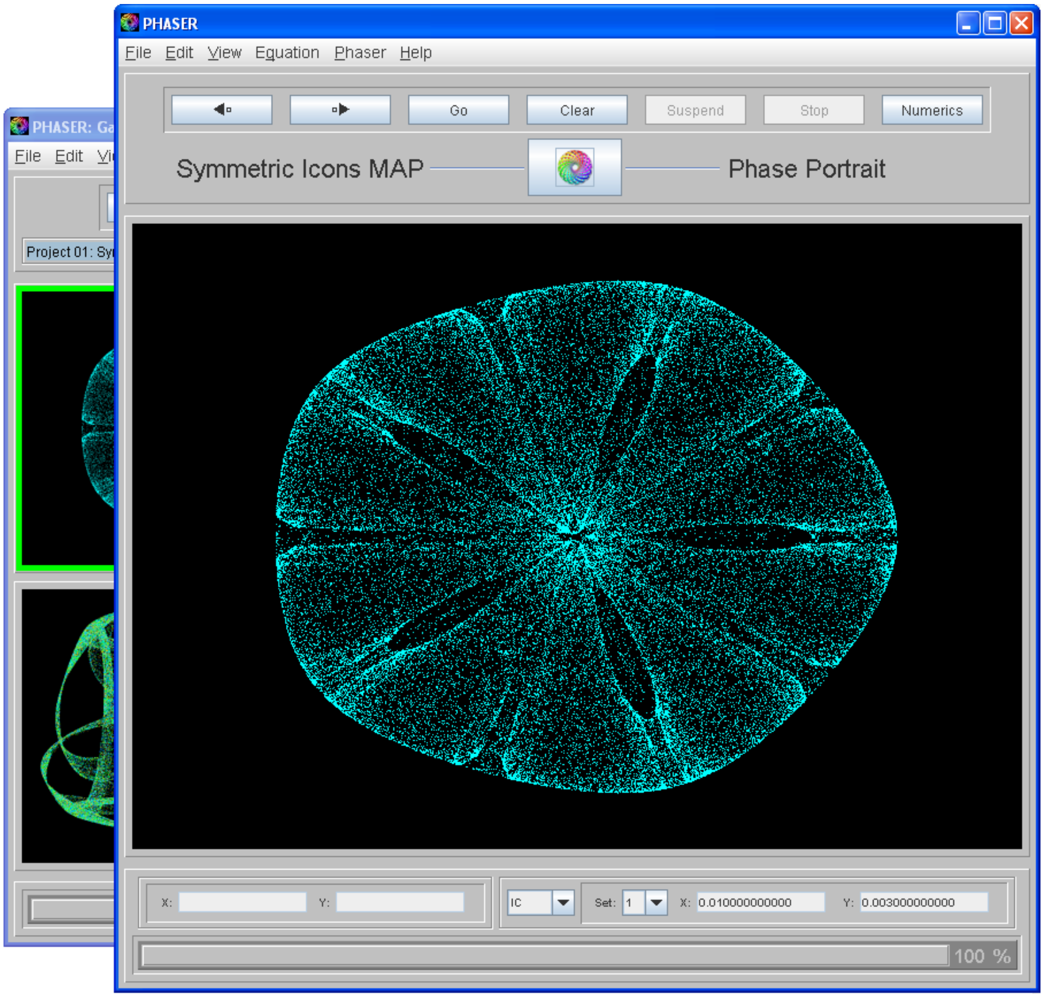
Results of Phaser simulations can be saved and shared over the Internet. If you have a reasonably fast connection, you can download and view a gallery of interesting images from our Web site by doing the following:

Phaser → Gallery	and from the spawned window, select
File → Load Gallery URL	in the spawned dialog box, type
http://www.phaser.com/projects/icons.pgf	and hit
Fetch	



Fetching a Phaser Gallery over the Internet

Now, click on a frame (boundary should turn red) and select **Phaserize** to see the small image loaded and computed in the Phaser main view.



Sanddollar as the attractor of a 2-D MAP

If you are curious which mathematical formula generates such a natural picture, select the menu item

Equation → View Current Equation Help

Interacting with Phaser

Here, general guidelines for interacting with Phaser are outlined in rather broad brush strokes. Detailed implementations of these guidelines are provided in Tutorials.

Selecting Equation: After launching Phaser, the first task is to select an equation from the permanent libraries, MAP Library or ODE Library, for numerical simulations. If the desired equation is not in the libraries, it can easily be entered as a Custom Equation using Phaser's Equation Editor. These options can be executed from the `Equation` menu located on the menu bar of the Main Application window.

Selecting View: The second task is to choose a graphical view appropriate for the intended study of the current equation. This can be accomplished from the `View` menu on the menu bar of the Main Application window. Alternatively, one can cycle through the views using the `<<` or `>>` buttons on the button bar of the Main Application window. Currently, there are five graphical views:

- `Phase Portrait View`: Two- or three-dimensional projections of orbits are plotted here. In three dimensions or higher, orbits can be rotated in real time. Flows, vector fields, and Poincaré maps can also be displayed.

- `Xi vs Time View`: Selected Xi variables vs. Time are plotted here. In dimension one, direction fields can be superimposed.

- `Bifurcation Diagram View`: One of the variables versus a parameter can be plotted to investigate possible bifurcations.

- `1-D Stair Stepper View`: Stair-step or cobweb diagrams of orbits of one-dimensional maps can be plotted in this view.

- `Xi Values View`: Computed numerical values of the Xi variables are tabulated here.

Numerics Editor: The third task is to configure the graphical display options for the current view and to fix the setting for numerical computations of solutions. These multitude of choices are made from `Numerics Editor` housed in the co-primary window. This window can be brought to the foreground by clicking on the `Numerics` button on the button bar of the Main Application window. Numerics Editor has the following five important buttons on its button bar:

- `Current View`: To configure the current graphical view; e.g. Window size. Each graphical view has its private set of options.

- `Initial Conditions`: To specify the values and colors of multiple initial conditions whose solutions are computed simultaneously and processed for the current view. Initial conditions can also be entered by clicking the mouse at the desired locations in the current graphical view.

- `Parameters`: To set the values of the parameters in the current equation.
- `Time`: To set the duration of computations of solutions.
- `Algorithms`: To select the current algorithm and its control settings to be used in numerical computations of solutions.

Local and Global Settings: The configuration set by `Current View` is local in the sense that it applies only to the current view and is independent of the configurations of the other views. Consequently, each view must be configured individually. The remaining items set using `Initial Conditions, Parameters, Time,` and `Algorithms` are global and used in all views. If a new equation is chosen, some of the global settings are set to their default values.

Apply: Choices made or the numbers typed into text fields in Numerics Editor do not take effect unless the `Apply` button of the Editor is clicked. APPLY!

Mouse: When the cursor is within a graphical view, the mouse can be used for several different actions, such as setting Initial Conditions (IC), Zoom In/Out, Pan, 3-D Rotations, or Flow. The drop-down choice widget in the center of the task bar is the mouse action selector.

Save, Restore, and Gallery: Obtaining an informative or pleasing image of a dynamical phenomenon can require extensive experimentation involving many settings of the graphical and numerical choices. The entire state of Phaser necessary to recreate an image can be saved using the `Save Current Project` entry of the `File` menu of the Main Application Window. Collections of images can be exhibited in the Gallery.

Help: The contents and the index of online Help are searchable. Detailed information about a particular Phaser function, or technical and scientific information about an equation in the Library can easily be located.

Tutorials

The sequence of lessons below are designed to guide the user, step by step, through some of the basic capabilities of Phaser. In these lessons, it is assumed that Phaser is in the start up configuration, and that values of all things are set to the defaults. Therefore, before starting each lesson, either Phaser should be restarted or Preferences should be set to System Defaults.

Lesson 1. First Steps: Computing e

A must Lesson for getting started.

Lesson 2. Xi vs. Time

Using the Logistic differential equation (ODE), how to display solutions on the Xi vs Time plane, drawing direction fields.

Lesson 3. Stair-Step Diagrams

Using the Logistic difference equation (MAP), how to display stair-step diagrams.

Lesson 4. Bifurcation Diagram of Logistic MAP

How to compute bifurcation diagram of this famous map. Zoom in and out of regions of the bifurcation diagram.

Lesson 5. Phaser Projects

How to save a current state of Phaser, reload it later, or share it over the net.

Lesson 6. The Gallery

How to create a gallery of live images and their animations.

Lesson 7. Custom Library: Entering your equations

How to enter or modify your own ODE or MAP into the Custom Library of Phaser. All done with an easy-to-use Equation Editor.

Lesson 8. Flow box

How to follow 5000 solutions, starting from a selected box, simultaneously.

Lesson 9. Cloning Current Phaser

How to spawn a new Phaser and run two copies simultaneously. An unforgettable example from chaotic numerics.

Lesson 10. Real-Time 3-D Graphics

How to twirl around 3-D orbits in real time.

Lesson 11. Poincaré Map in 3-D

How to slice an orbit in 3-D with any plane, and compute the points of intersection on the plane in one or both directions.

Lesson 12. Poincaré Map of Forced Duffing ODE

How to compute Poincaré maps of periodically forced second-order oscillators. Periodic and non-periodic orbits of Forced Duffing are used as examples.

Lesson 13. Sequencing Slides

How to generate a sequence of slides. As an example, a sequence of fractal curves with dimensions from 1 to 1.9 in increments of 0.1 are generated.

Lesson 1. First Steps: Calculating e

We commence our lessons with Phaser using a simple but delightful example. Let us consider the initial-value problem

$$\dot{x} = x, \quad x(0) = 1$$

and determine numerically x(1.0), the value of the solution at t = 1.0. The solution of this elementary problem is the exponential function $x(t) = e^t$. Thus the desired value x(1) is the famous number *e*. The value of *e* can be found with a decent pocket calculator to be

$$e = 2.718281828459045... \quad .$$

In this lesson, we will numerically integrate the initial-value problem above, to compute the value of *e*. This may appear to be pedantic but having this number in our possession will permit us to test the accuracy of our numerical integrator on this initial-value problem.

»> `Equation -> ODE Library -> Cubic 1-D ODE:`

The first task is to specify the equation we would like to study. To get our differential equation, click on the `Equation` menu on the menu bar of the Main Application window, and select `ODE Library.` From the list of ODEs (Ordinary Differential Equations), select `Cubic 1-D ODE.` Notice that the banner is now updated to display the name of the current equation. `Cubic 1-D ODE` is a general purpose one-dimensional differential equation, but what does it look like?

»> `Equation -> View Current Equation Text:`

From `Equation` menu, select the `View Current Equation Text.` A small window containing the text of `Cubic 1-D ODE` should pop. In the text

$$\dot{x}_1 = a + b\,x_1 + c\,x_1^2 + d\,x_1^3,$$

a, b, c, and d are changeable parameters and x_1 is the variable. We will momentarily set the appropriate parameter values to get our differential equation.

»> `Numerics:`

In preparation for setting the desired parameter values, click on the `Numerics` button on the button bar of the Main Application Window. This displays the Numerics co-primary window.

»> `Parameters:`

Click on the `Parameters` button on the button bar of Numerics Editor. The four parameters and their current default values are now visible. We need to set a=0, b=1, c=0, and d=0 to obtain our differential equation from `Cubic 1-D ODE.` Now, click in the text field containing the value of b, which currently is -1.0, and change it to 1.0. There are several convenient ways to change text fields: click after a character you would like to change and use the backspace key, or double click anywhere in the field to highlight the field and type your new characters. We can leave the other parameter values as they are. Notice the `(modified)` token on the banner and Status Field of Numerics Editor signaling the fact that a current setting has been modified.

»> `Apply:`

Click on the `Apply` action button on the lower-right corner of Numerics Editor for these new parameter values to take effect. Notice that the `(modified)` token on the banner of Numerics Editor has disappeared, signaling that the new values of the parameters have been registered. Status Field in the lower-left corner of Numerics Editor now indicates `Ready.`

»> `Initial Conditions:`

Click the `Initial Conditions` button on the button bar of Numerics Editor. Notice that there is one set of initial conditions activated with the default values t = 0.0 and x_1 = 1.0. These values happen to be what we want; so we will not touch them.

»> `Time:`

Click the `Time` button on the button bar of Numerics Editor. Notice that the `Start Plotting` time is set to 0.0, and the `Stop Plotting` time is set to 20.0. Time is relative; that is, the solution from the initial time will be computed and displayed for 20 time units in the forward direction. Since we want to see the value of the solution at t = 1.0, we set `Stop Plotting` to 1.0, and APPLY.

»> `Main Application Window:`

Click on the title bar of the Main Application window to bring the main window to the foreground. You can also click on the `Phaser` action button on the lower right-hand corner of the Numerics Editor to accomplish the same task.

»> `View -> Xi Values:`

From the `View` menu of the Main Application window select `Xi Values`. Now the current view is set to `Xi Values` which is the appropriate view for tabulating numerical values of solutions.

»> `Go:`

After this extensive preparation, now click on the `Go` button on the button bar of the Main Application window to see the results. The last line is the computed value of x_1 at time 1, that is, 2.7182818285E+000. Notice that this output is in scientific notation, which is the output format used in Phaser. Also, it is interesting to observe that the last digit has been rounded up by the compiler.

Lesson 2. Xi vs. Time

In this lesson, we will learn how to customize the `Xi vs. Time` view, while investigating the dynamics of a single population modeled with the logistic differential equation

$$\dot{x} = x - x^2.$$

Here x(t) represents the population density as a function of time. The qualitative dynamics of this differential equation are rather simple. The equation has two equilibrium states: x(t) = 0.0 and x(t) = 1.0. The first equilibrium is not of much ecological interest as it corresponds to zero population density. The second equilibrium, however, is of paramount importance because the solution from any positive initial condition approaches 1.0 as time increases. This positive limiting equilibrium density is the carrying capacity of the population. We shall see now the approach to the carrying capacity as time goes on.

»> `Equation -> ODE Library -> Cubic 1-D ODE:`

To get our differential equation, click on the `Equation` menu on the menu bar and select the `ODE Library`. From the list of ODEs, select `Cubic 1-D ODE`.

»> `Equation -> View Current Equation Text:`

From the `Equation` menu, select `View Current Equation Text.` A small window containing the text of `Cubic 1-D ODE` should pop. In the text

$$\dot{x}_1 = a + b\, x_1 + c\, x_1^2 + d\, x_1^3,$$

a, b, c, and d are changeable parameters and x_1 is the variable. We will next adjust the parameter values to set our logistic differential equation.

»> `Numerics:`

In preparation for setting the desired parameter values, click on the `Numerics` button on the button bar of the Main Application Window. This displays the Numerics co-primary window.

»> `Parameters:`

Click on the `Parameters` button on the button bar of the Numerics Editor. The four parameters and their current default values are now visible. We need to set a = 0, b = 1, c = -1, and d = 0 to obtain our logistic differential equation from `Cubic 1-D ODE.` Now, click in the text field containing the value of b and change it to 1.0. Next, click in the text field containing the value of c and change it to -1.0. Notice the `(modified)` token on the banner and the Status Field of the Numerics Editor.

»> `Apply:`

Click on the `Apply` button on the lower right corner of the Numerics Editor for these new parameter values to take effect. Notice that the `(modified)` token on the banner of the Numerics Editor has disappeared.

»> `Initial Conditions:`

Click the `Initial Conditions` button on the button bar of the Numerics Editor. Notice that there is one set of initial conditions activated with the default values `t = 0.0` and `x1 = 1.0`. These values happen to be the equilibrium state corresponding to the carrying capacity. Let us add two more initial conditions.

»> `Set 2` and `Set 3:`

To activate the second set of Initial Conditions, click on the left-most button next to `Set 2.` This should highlight the first two text fields, the first of which is the initial time and the second of which is the initial x1 value. Change the value of x1 to 0.1. In `Set 3` set x1 to 1.5. Apply.

»> `Phaser:`

Click on the `Phaser` action button to return to the Main Application Window.

»> `View -> Xi vs Time:`

Return to the Main Application Window by clicking on its title bar. From the `View` menu of select `Xi vs Time.` Now the current view is set to `Xi vs Time`. Vertical Axis will house the x1 variable and the horizontal axis will house time.

»> `Go:`

Now press `Go` and watch the three solutions being plotted simultaneously.

This picture is adequate, but we can do better. Let us now return to the Numerics Editor and fine tune the `Xi vs Time` view.

»> `Current View -> Window Size:`

Click on the `Current View` button and from the control tree choose `Window Size`. Set X-Min = -1, X-Max = 5, Y-Min = -1, and Y-Max = 2. Apply. Notice that the Xi vs Time view is cleared and Axes and Grids are redrawn.

»> `Axes and Grids:`

From the control tree choose `Axes and Grids`, then set the last two text fields, `X- and Y-Grid Gap Values,` to 0.2. Apply. Grids are now redrawn with the new gap sizes.

»> `Graphics:`

Click on the `Graphics` node on the control tree. The first entry on the right is `Graph Point Size` which sets the size of the points used in the plotting of the solutions. Click on the drop-down widget (small bar to the left of entry) and from the list choose 3.

»> `Direction Field:`

From the control tree choose `Direction Field` and press the left-most small square on the first line. Apply. Now, along with the solutions, the Direction Field will be drawn at grid points when we press Go.

»> `Time:`

Click the `Time` button on the button bar of the Numerics Editor. Set `Start Plotting` to -6.0, and the `Stop Plotting` time to 6.0. The solutions will be computed backwards for 6 time units and then computed forwards for another 6 time units. Apply.

»> `Go:`

Return to the Main Application window and press `Go` to see the result. Messages from the current numerical algorithm used to generate solutions appear in the `Numerics Console` co-primary window. Notice that the solution starting from Initial Condition Set 3, which blows up in backward time, has been stopped by the Dormand-Prince 5(4) algorithm as it cannot meet the set error tolerances.

»> `Finale:`

Press `Clear.` Click the mouse in about a dozen different locations inside the Xi vs Time view and press `Go.`

Lesson 3. Stair-Step Diagrams

Orbits of one-dimensional difference equations can be best followed using a geometric method known as Stair-Step, or cobweb, diagram. In this lesson, we will learn how to use this method while exploring the dynamics of decidedly the most famous one-dimensional difference equation, the `Logistic MAP`

$$x_1 \mapsto ax_1(1.0 - x_1)$$

Stair-step diagrams: Here is how a stair-step diagram of a solution of a difference equation x_1 -> f (x_1) is computed. We first draw the 45-degree line through the origin and the graph of f. Then, for an initial value of x_1 on the horizontal axis, we find `next` x_1 by drawing the vertical line segment from the initial point until it meets the graph of f. Now, to transfer the vertical coordinate of this point of intersection to the horizontal axis, we draw the horizontal line segment from the point of intersection to the 45-degree line. Now, from the 45-degree line we draw the vertical line segment to the horizontal axis. Now that we are back on the horizontal axis, we repeat the previous step over again to find the next point in the orbit, and so on.

Ranges: The relevant dynamics of Logistic MAP take place for the values of the parameter *a* between 0.0 and 4.0, and the x_1 variable between 0.0 and 1.0. The choice of the initial condition on the unit interval is not important since almost all orbits eventually behave about the same. The value of the parameter *a*, however, is crucial for the asymptotic dynamics, as we will see momentarily.

»> `Equation -> MAP Library -> Logistic MAP:`

From the Equation menu, select MAP Library and then pick `Logistic MAP`. Now the current equation is Logistic MAP as seen on the banner.

»> `View -> 1-D Stair Stepper:`

From the `View` menu, select `1-D Stair Stepper` to make this view the current view. Again, note the current equation and the current view names on the banner.

»> `Numerics:`

Click on the Numerics button on the button bar to open the Numerics Editor.

»> `Window Size:`

Notice that the Numerics Editor automatically displays the tree of the Current View. From the tree, select `Window Size`. By clicking on the text fields, set `X-Min` = -0.1, `X-Max` = 1.1, `Y-Min` = -0.1, and `Y-Max` = 1.1. Apply.

»> `Parameters:`

Click on the `Parameters` button on the button bar of the Numerics Editor. Set the value of the parameter a = 2.8. Apply.

»> `Go:`

Click on the title bar of the Main Application Window to bring the Current View to the foreground. Press `Go` to see the Stair Step diagram drawn.

»> `Current View -> Graphics -> Graph Plotting Repaint Interval Delay (ms):`

To see how this picture really developed, let us recreate it in slow motion. For this purpose, return to the Numerics Editor and click on the `Current View` button and choose `Graphics` from the control tree. Set the `Graph Plotting Repaint Interval Delay (ms)` to 1000. Apply. Phaser will pause 1000 milliseconds between two plotted points. Depending on the speed of your computer, you might want to adjust the delay time.

»> `Clear` and `Go:`

Return to the Main Application Window and hit `Clear` and `Go`. The graph of the logistic function should appear immediately, but the successive points on the computed orbit should appear at a speed that allows you to follow the dynamics: The orbit converges to a fixed point. Notice that the fixed point is the intersection of the 45-degree line with the graph of the function.

»> `Graph Plotting Repaint Interval Delay (ms):`

Set `Graph Plotting Repaint Interval Delay (ms)` to 1, to speed things up again. Apply.

»> `Parameters:`

Set a = 3.4. Apply.

»> `Clear` and `Go:`

The orbit no longer approaches the fixed point. Where does it go?

»> `Time:`

Return to the Numerics Editor and click on the `Time` button. To discard the transient (early) part of the orbit, set `Start Plotting` to 333 and `Stop Plotting` 3333. Apply.

»> `Clear` and `Go:`

The orbit approaches a periodic orbit of period 2. This qualitative dynamical change where a fixed point gives up its stability to a periodic orbit of period 2 is called a *period-doubling bifurcation.*

»> `Parameters:`

Set a = 3.55. Apply.

»> `Clear` and `Go:`

You should now see a periodic orbit of period 4. This is another period-doubling bifurcation.

»> `Parameters:`

Set a = 4.0. Apply.

»> `Clear` and `Go:`

You should now see the orbit nearly fill the view - Chaos.

Lesson 4. Bifurcation Diagram of Logistic MAP

In this lesson, we will show how to compute the bifurcation diagram of the Logistic MAP

$$x_1 \mapsto a x_1 (1.0 - x_1)$$

This diagram will depict one of the most famous routes to chaos through successive period-doubling bifurcations as the parameter `a` is varied.

Bifurcation diagram: Here is how a bifurcation diagram is computed. One of the parameters in the definition of the current equation is assigned to the X-Axis, and one of the variables x_i is assigned to the Y-Axis. Then the parameter range, determined by X-Min to X-Max, is divided into a number of specified subdivisions. For each parameter value in this subdivision, the orbits are computed and plotted using the current settings of Initial Conditions, Time, and Algorithm.

»> `Equation -> MAP Library -> Logistic MAP:`

From the Equation menu, select MAP Library and then pick `Logistic MAP.`

»> `View -> Bifurcation Diagram:`

From the `View` menu, select `Bifurcation Diagram` to make this view the current view. Notice the current equation and the current view names on the banner.

»> `Numerics:`

Click on the Numerics button on the button bar to open the Numerics Editor. Notice that the Numerics Editor automatically displays the tree of the Current View.

»> `Bifurcation Diagram:`

The root of the control tree of the Current View is also called `Bifurcation Diagram`. Let us examine the contents of the root. Notice that the parameter *a* is assigned to the X-axis and the variable x1 is assigned to the Y-axis. Since there is only one parameter and the equation is one dimensional, there are no choices to be made. `Parameter Sample Size` is set to 250; thus, there will be 250 parameter subdivisions in the computation of the bifurcation diagram.

»> `Window Size:`

From the tree, select `Window Size`. By clicking on the text fields, set `X-Min` = -0.1, `X-Max` = 4.1, `Y-Min` = -0.1, and `Y-Max` = 1.1. Apply. Thus, 250 equally spaced values of the parameter *a* in the range from -0.1 to 4.1 will be used.

»> `Initial Conditions:`

Click on the Initial Conditions button. There is only one active set which is 0.2 at time 0.0. This is fine, since almost any initial condition between 0.0 and 1.0 gives the same bifurcation diagram.

»> `Time:`

Click on `Time` on the button bar and set `Start Plotting` to 999 and `Stop Plotting` to 2222. Apply. With these numbers, we discard the first 999 iterates and record the limiting behavior of solutions. Why these numbers? Well, they are easy to type with one finger.

»> `Go:`

Now, click on the title bar of the Main Application Window to bring it to the foreground and press `Go`. Bifurcation diagram should gradually come to life.

»> `Parameter Sample Size:`

Let us increase the resolution of the bifurcation diagram. Return to the Numerics Editor and click on the `Current View` button. Then click on the root of the tree, `Bifurcation Diagram`. Now press the drop-down choice widget to the left of `Parameter Sample Size` and set it to 500. Apply.

»> `Clear` and `Go:`

A finer bifurcation diagram.

Period-doubling cascades: It is visible in the bifurcation diagram that for the parameter values until 3.0, the orbit is attracted to a fixed point. The location of the fixed point changes as the parameter is varied. When the parameter is increased past 3.0, the fixed point gives up its stability to an asymptotically stable periodic orbit of period 2. This bifurcation is known as a *period-doubling bifurcation.* As the parameter is increased further, the orbit undergoes successive period-doubling bifurcations and eventually becomes non-periodic.

Period three: Notice that there are also some dark gaps in the bifurcation diagram. For example, the largest gap contains a periodic orbit of period 3. Let us now enlarge a portion of the bifurcation diagram to reveal some of its fine structure.

»> `IC:`

The choice button in the center of the Tool Bar is a mouse action selector. `IC` indicates that the current mouse function is for setting Initial Conditions. Click on the drop-down widget next to `IC` in the middle of the Tool bar and select `ZOOM`.

»> `ZOOM:`

In the ZOOM mode, the mouse can be used to zoom into a selected region of the graphical view or to zoom out by a chosen factor. The default setting is to zoom in and we shall leave it unchanged. Notice that the cursor inside the view is changed. Now, position the cursor near X = 3.75 and Y = 1.0. You can see the coordinates of the current cursor location in the left portion of the Tool Bar. Click the left mouse button and release it. Next, move the cursor to the location X = 3.9 and Y = 0.09. Notice the rubber band marking the selected rectangle. Click the left mouse button again. The enlargement of the selected region of the Bifurcation Diagram should be underway automatically. Notice the period-three orbit in the middle.

»> `ZOOM:`

The period-three orbit appears to undergo a sequence of period-doubling bifurcations as well. To see this phenomenon more clearly, let us zoom into the middle portion of the bifurcation diagram. Now, position the cursor near X = 3.838 and Y = 0.58; click the left mouse button and release it. Next, move the cursor to the location X = 3.857 and Y = 0.39; click the left mouse button again. The enlargement of the selected region of the Bifurcation Diagram should be underway automatically. It is remarkable that this bifurcation diagram looks much like the first one we plotted.

Lesson 5. Phaser Projects

During simulations, current state of Phaser (numerical and graphical settings of all things) can be saved to a small user-specified file by simply using `Save Current Project` button on the `File` menu of the Main Application Window. A previously saved Phaser Project file can be reloaded into Phaser using `Load Phaser Project` button on the same menu. This action restores the current state of Phaser to that of the saved project.

A more important function provided in Phaser is the ability to load previously saved Phaser Project files over the net using `Load Phaser Project URL` button on the `File` menu of the Main Application Window. This action enables you to `Fetch` the desired Phaser Project file using http protocol: `http://< file location >.`

In this lesson, we will load a Phaser Project file modeling reversals of earth's magnetic field from the Web site of Phaser. We assume that you are connected to the internet.

»> `File -> Load Phaser Project URL:`

From the the `File` menu of the Main Application Window, choose `Load Phaser Project URL`. A dialog box should pop.

»> `http://www.phaser.com/projects/rikitake3.ppf`

Click in the text field of the dialog box and type the URL address above. We have deposited this project file `rikitake3.ppf` at this location.

»> `Fetch:`

Now hit the `Fetch` button. The dialog box should disappear and the project should load. As soon as the loading is finished, two orbits of a three-dimensional ODE named `Rikitake Disc Dynamos ODE` should begin to appear.

»> `Equation -> View Current Equation:`

To see the text of the equations, select `View Current Equation` entry from the `Equation` menu of the Main Application Window. These equations describe the motion of two coupled disc dynamos. The variables x_1 and x_2 are (scaled) currents of the two dynamos. The parameter `m` is a function of the self-inductance, resistance, mutual inductance, and the driving torque of the dynamos, and the parameter `a` is the (scaled) difference of the angular velocities of the dynamos.

»> `File -> Open Phaser Project Notes :`

A Phaser Project File (.ppf) comes with a text file containing notes about the current project. To view the Notes for the current project, from the `File` menu, select `Open Phaser Project Notes`.

»> `3-D:`

Notice that the mouse is in 3-D mode as indicated on the tool bar. While pressing down the left mouse button, you can move the mouse around to rotate the three-dimensional orbits to get more informative pictures.

»> `View -> Xi vs Time:`

From the View menu of the Main Application Window, select `Xi vs Time`. Press Go to see the x_1 vs Time plots of the two solutions. This is a revealing picture. Since x_1 is the current through the first dynamo, a change in sign of x_1 corresponds to a reversal of the current and hence a reversal in the polarity of the magnetic field. Notice the non-periodic - chaotic - reversals of sign of x_1.

»> `Equation -> Import Custom Equation:`

If you are interested in exploring `Rikitake Disc Dynamos ODE` further at a later time, you can import the equations into your private Custom library. Simply, choose the entry `Import Custom Equation` from the Equation menu. (If the equation is already in the Custom Library, the menu entry will be grayed out.)

»> `File -> Save Current Project As...:`

You can also save the current imported project locally on your computer by selecting `Save Current Project As...` from the File menu of the Main Application Window. Now that the imported project is safely saved, you can change the settings to explore the dynamics. For example, set the parameter `m = 0.5` to see periodic oscillations of the currents. Now, you can return to the Phase Portrait view, and Clear and Go, to see the three-dimensional picture of the periodic oscillations.

Further mathematical details on the Rikitake two-disk dynamo system can be found in Cook and Roberts [1970] and Robbins [1977]. Information about dynamos and earth's magnetism is available at `http://www-spof.gsfc.nasa.gov/earthmag/dynamos.htm`

Lesson 6. The Gallery

In this lesson, we will investigate certain bifurcations in the dynamics of the area-preserving `Cremona MAP`

$$x_1 \mapsto x_1 \cos a - (x_2 - x_1^2) \sin a$$
$$x_2 \mapsto x_1 \sin a + (x_2 - x_1^2) \cos a$$

by plotting phase portraits as we vary a parameter. We will organize these phase portraits into a gallery of iconic images and manipulate them.

»> `Equation -> MAP Library -> Cremona MAP:`

From `Equation` menu, select `MAP Library`. From the list, select `Cremona MAP`.

»> `Equation -> Load Equation Defaults:`

Again from `Equation` menu, select `Load Equation Defaults.` Now all settings will be loaded from a system file which contains values yielding an informative picture. In particular, the equation parameter is set to `a = 1.32843.` You should now see a phase portrait of Cremona MAP consisting of about a dozen initial conditions iterated 1500 times.

»> `View -> Send View to Gallery:`

From `View` menu of the Main Application window, select `Send View to Gallery.` The contents and the associated settings that are necessary to recreate this phase portrait will be sent to the Gallery.

»> `Phaser -> Gallery:`

To see where the phase portrait is deposited, select `Gallery` from `Phaser` menu of the Main Application window. The co-primary window Gallery should pop. Gallery currently consists of four frames and the phase portrait is deposited in the first frame.

»> `Parameter:`

From Numerics Editor, set the parameter value to `a = 1.56.` Apply, return to Main Application window, and Clear and Go. Now, you should see the phase portrait of Cremona MAP for the new value of the parameter.

»> `View -> Send View to Gallery:`

To deposit the new phase portrait into Gallery, select `Send View to Gallery.` Notice now that the new phase portrait is deposited in the next available frame.

»> `Two more frames:`

Next, set the parameter to `a = 1.92.` Apply, Clear and Go, and Send View to Gallery. Repeat with `a = 2.0.` All four phase portraits should now be deposited in Gallery.

»> `Send to a Selected Frame:`

Let us now change the contents of a selected frame. First, select the second frame (1, 2), making sure that its boundary is red. Return to Numerics Editor and change the parameter value to `a = 1.61`. Apply and Clear and Go. Then click on `Send View to Gallery.` Now, the contents of the Phase Portrait view will be resent to the second frame. Unselect the second frame by clicking in it.

»> `Animate:`

Now, click on the `Animate` button on the button bar of Gallery. The frames should be played sequentially in the Main Application window.

»> `File -> Open Gallery Project Notes:`

From the `File` menu of the Gallery co-primary window, select `Open Gallery Project Notes`. In the spawned small window, you can type in a description of the Gallery you have just created. When the current Gallery is saved, these notes are saved along with the images.

»> `File -> Save Current Gallery:`

Finally, you can save the current Gallery in a file using `Save Current Gallery` entry on the `File` menu of the Gallery co-primary window.

Lesson 7. Custom Library: Entering new equations

Phaser provides an easy-to-use `Equation Editor` with a graphical interface for adding new equations to the library. Once entered successfully, a new equation is deposited into the Custom Equation Library of PHASER and is indistinguishable, as far as the user is concerned, from the Permanent Library equations. Previously added user-defined equations can also easily be modified or deleted.

Phaser is designed for first-order ordinary differential (ODE) or difference (MAP) equations. A higher-order equation should first be converted to a system of first-order equations.

In this lesson, we will use the pendulum equations to illustrate the process of entering a new equation into PHASER:

x1' = x2
x2' = -(g / l) sin(x1) - (c / (l m)) x2

where g, l, c, and m are parameters.

»> `Equation -> Add Custom Equation:`

From the `Equation` menu on the menu bar of the Main Application window, choose `Add Custom Equation.` The new window that pops is the `Equation Editor` co-primary window for adding a new equation. In the remainder of this tutorial, we will populate the requisite fields in the `Equation Editor`.

»> `Equation Name:`

Each new equation must be named. Let us call our new equation `My Pendulum ODE.` Click on the text field and type this name. By convention, we recommend the usage of the suffix ODE to tag an ordinary differential equation, and MAP to tag a difference equation.

»> `Equation Description (optional):`

This field holds an optional descriptive sentence of the equation and is not used by Phaser. Click on the text field and type, for example, `Practicing how to enter a new equation.`

»> `Equation Type:`

Library equations are sorted into two types: ODE and MAP. Since `My Pendulum ODE` is ODE, we need not change the type. If you were entering a difference equation, from the drop-down widget, you would choose MAP.

»> `Default Algorithm:`

Each equation is provided with a default algorithm. The current default algorithm for ODE type is `Dormand-Prince 5(4),` which is fine for our equation. The drop-down widget contains other ODE algorithms.

»> `Parameter Name:`

Our equation has four parameters: `g, l, c,` and `m.` We need to tell Equation Editor about these parameters and their default values. Any single lower- or upper-case letter, except `t`, `x`, and `E`, which are reserved for time, space, and the famous constant, can be used as a parameter. Click on the text field and type `g.`

»> `Parameter Value:`

Click on the text field and enter `1.0`. This is the default value of the parameter `g.`

»> `Add Parameter:`

Click on this button to add the parameter `g` and its default value. Now the `Parameter List` field should show `g = 1.0.`

»> `Repeat three steps:`

Now repeat the previous three steps with the remaining three parameters until Parameter List shows `g = 1.0`, `l = 1.0`, `c = 0.0`, `m = 1.0.`

»> `Parameter Editing:`

To change the name or the default value of a parameter, click on it in `Parameter List` and change things as desired; re-add parameter. To remove a parameter from Parameter List, highlight the desired parameter by clicking on it and then push the `Remove Parameter` button.

»> `x1' = :`

Click on the text field and type `x2`. This is the right-hand side of the first equation.

»> `x1 = :`

This is the default initial value of the variable x1. Click on the text field and type `1.5.` The initial displacement is 1.5.

»> `x2' = :`

To activate this field, first click on the button to the left of x2' =. This should highlight the text field for x2'. Now, click on the text field and type `-(g / l) * sin( x1 ) - (c / (l * m) ) * x2.`

»> `x2 = :`

This is the default initial value of the variable x2. We can leave it alone - zero initial velocity.

»> `Add:`

Finally, click on the `Add` button to add your new equation to the Custom Library of Phaser. At this point, Phaser attempts to parse the expressions for x1' and x2'. Barring some typographical errors, Equation Editor should report in Status Field that expressions are parsed and the new equation is added. If it fails to parse, correct the offending errors and `Add` again.

»> `Close:`

Now, you can safely `Close` Equation Editor from the File menu of the Editor.

»> `Viewing Current Equation:`

From the `Equation` menu of the Main Application window select `Custom Equation Library.` Click on `My Pendulum ODE` to make it the current equation. Finally, from the `Equation` menu, select `View Current Equation.` A small window containing the text of `My Pendulum ODE` should pop. Examine the text carefully before embarking on computations. You can correct any typographical errors using the `Modify Custom Equation` entry of the `Equation` menu.

Parser Syntax: x1, x2, ..., x24 are special tokens and must be written exactly and without spaces. For detailed specifications on the operators, operands, and functions supported by the parser, you can consult `Phaser: Equation Editor` in the Table of Contents of Phaser Help. For your convenience, this page can be directly accessed by choosing `Help Contents` from the Help menu of Equation Editor.

Speed: A user-entered equation runs approximately at half the speed of its compiled counterpart. If you like, now you can check out the speed and the accuracy of Phaser's parser/evaluator by running a computation with the compiled equation `Pendulum ODE` and its parsed counterpart `My Pendulum ODE`.

Lesson 8. Flow box

In this lesson we will investigate the dynamics of `Cremona MAP`

$$x_1 \mapsto x_1 \cos a - (x_2 - x_1^2) \sin a$$
$$x_2 \mapsto x_1 \sin a + (x_2 - x_1^2) \cos a$$

by following the fates of 10,000 solutions starting from randomly chosen initial conditions in a selected box.

Cremona MAP is a two-dimensional area-preserving map. Near fixed or periodic points of an area-preserving map, two distinct dynamical behaviors prevail:

- **elliptic:** Near an elliptic point, solutions usually rotate along closed invariant curves. A box about an elliptic point rotates, albeit distorted, as it moves under the iterations of the map.
- **hyperbolic:** Near hyperbolic points, some solutions move towards the point while others move away. Consequently, a box about a hyperbolic point is stretched along one direction while being contracted along another. Since the area of the box must be preserved, the box can undergo remarkable topological changes under the iterations of the map.

In this lesson, we will explore these two types of dynamical behavior in `Cremona MAP`.

»> `Equation -> MAP Library -> Cremona MAP:`
From the `Equation` menu on the menu bar, select `MAP Library`. From the list of MAPs, select `Cremona MAP.` This is an are-preserving map on the plane.

»> `Equation -> Load Equation Defaults:`
From the `Equation` menu on the menu bar, select `Load Equation Defaults`. You should see phase portrait of Cremona MAP.

»> `Numerics:`
Click on the `Numerics` button on the button bar to open the Numerics Editor.

»> `Graph Plotting Repaint Interval:`
From the drop-down choice widget next to `Graph Plotting Repaint Interval,` choose 1. Apply. Now, the graphical view will be updated after each iteration.

»> `Graph Repaint Interval Delay (ms):`

To see the evolution of the pictures we are about to plot, it is necessary to introduce a time delay into plotting. Depending on the speed of your computer, you should adjust this setting (an integer) to get the desired effect. For example, set `Graph Repaint Interval Delay (ms)` to 500. Apply.

»> `Stop Plotting:`

Select the `Time` button on the button bar of the Numerics Editor, and set `Stop Plotting` to 100.

»> `Phaser:`

Click on the action button `Phaser` to return to the Main Application window.

»> `Flow:`

From the drop-down choice widget next to IC on the tool bar of the Main Application window, select FLOW. Now the mouse function will be to select a flow box.

»> `Number of random initial conditions:`

From the drop-down choice widget next to `Number of random initial value points in the flow box` select 10,000. (For a slow CPU, this number should be smaller.)

»> `Flow box coordinates (elliptic):`

Move the cursor to coordinates about (-0.15, 0.15) and click and release left mouse button. Move the cursor to about (0.15, -0.15) and click and release left mouse button. Now, from 10,000 randomly chosen initial conditions in this square, solutions will be computed and displayed at each iteration. You should see the image of a red square under iterations. Since this square is in the region near the origin dominated by elliptic dynamics, the square should rotate under the iterations in a somewhat distorted manner.

»> `Clear:`

While the picture is evolving, occasionally press `Clear` button on the button bar of the Main Application window. You should see the current state of the original square. (You can `Stop` the computations any time.)

»> `New flow box coordinates (hyperbolic):`

Move the cursor to coordinates about (-0.42, 0.42) and click and release left mouse button. Move the cursor to about (-0.30, 0.30) and click and release left mouse button. Now, from 10,000 randomly chosen initial conditions in this box, solutions will be computed and displayed after each iteration. You should see the image of a red box under iterations.

Since this box is in the region dominated by hyperbolic dynamics of a period-5 point, the box should rotate under the iterations and visit the vicinity of five periodic points. After a few iterations, the original box should start spreading into a complicated geometry. Indeed, the dynamics near this hyperbolic periodic orbit of saddle type is chaotic.

»> `Clear:`

While the picture is evolving, occasionally press `Clear` button on the button bar of the Main

Application Window. You should see the dramatic current state of the original square.

Suggestions for further explorations: Cremona MAP undergoes interesting bifurcations as the parameter a is varied. In the Lesson on Gallery, we explore the dynamics as the parameter a passes through 2pi/3 (approximately 2.0943951).

Another nice use of Flow is to explore the phase portrait of Pendulum ODE while changing the friction coefficient from c = 0 to c = 0.5.

Lesson 9. Cloning Current Phaser

In this lesson, we will iterate the `Logistic MAP`

$x \to 4x(1 - x)$

and the `Logistic III MAP`

$x \to 4x - 4x^2$

100 times using the Initial Condition $x = 0.2$. Notice that the these two maps are mathematically the same. To the computer, however, they are rather different, as the numbers will reveal. For ease of comparison of numbers, we will clone Phaser and carry out computations using two distinct copies of Phaser.

We first customize Phaser to compute the desired numbers with the `Logistic MAP`.

»> `View:`

From the `View` menu of the Main Application Window select `Xi Values.` Now the current view is set to `Xi Values` which is the appropriate view for tabulating numerical values of solutions.

»> `Equation:`

From the `Equation` menu on the menu bar, select `MAP Library`. From the list of MAPs, select `Logistic MAP`.

»> `Viewing Current Equation:`

From the `Equation` menu, select the last entry `View Current Equation`. A small window containing the text of `Logistic MAP`, $x \to ax(1 - x)$ should pop.

»> `Numerics Editor:`

Click on the title bar of the Numerics Editor to bring it to the foreground.

»> `Parameters:`

Click on the `Parameters` button on the button bar of the Numerics Editor. Set $a = 4.0$ and Apply.

»> `Initial Conditions:`

Click on the `Initial Conditions` button on the button bar of the Numerics Editor. Notice that there is one set of initial conditions activated with the default values $t = 0.0$ and $x_1 = 0.2$. These values happen to be what we want; so we will not touch them.

»> `Time:`

Click on the `Time` button on the button bar of the Numerics Editor and set `Stop Plotting` to

100, and Apply.

»> `Current View:`

Click on `Current View` on the button bar of Numerics Editor and set `# of Last Xi Values to View` to 100. Apply.

»> `Phaser:`

Click on `Phaser` action button to bring the Main Application Window to the foreground.

»> `Go:`

Now click on the `Go` button on the button bar of the Main Application Window to see the results. You can use the slider on the right to browse through the 100 iterates.

Now, we clone an additional copy of Phaser with the settings above. The clone is labeled PHASER [1], and we will use the label [1] to tag the actions in this copy.

»> `Clone Current Phaser:`

From `File` menu of the Main Application Window, select `Clone Current Phaser`. A new copy of Phaser with all the current settings, such as Equation, Parameters, Current View, etc., should spawn. Notice that the Clone Phaser is tagged with `PHASER [1]` on its title bar. Click on `Numerics` on the button bar of `PHASER [1]` to spawn `Numerics Editor [1]`.

»> `Equation [1]:`

From the `Equation` menu of the Clone PHASER [1], select MAP Library and then `Logistic III MAP`.

»> `Numerics [1]:`

Click on the `Numerics` button of PHASER [1] to bring the Numerics Editor of the clone.

»> `Parameters [1]:`

Click on the `Parameters` button on the button bar of the Numerics Editor [1]. Notice that $a = 4.0$.

»> `Initial Conditions [1]:`

Click on the `Initial Conditions` button on the button bar of the Numerics Editor [1]. Notice that there is one set of initial conditions activated with the default values $t = 0.0$ and $x_1 = 0.2$.

»> `Go [1]:`

Click on the `Go` button on the button bar of PHASER [1] to see the 100 iterates of the initial condition $x_1 = 0.2$ under the `Logistic III MAP`.

Now, using the sliders of the Xi-Values views of the two Phasers, examine the numbers and try to determine when they begin to diverge. Do you prefer one set of numbers over the other?

Lesson 10. Real-time 3-D Graphics

When the current equation is of dimension three or higher, the Phase Portrait view is configured for 3-D Graphics functionality. Some of the key points to note are:

- If the current equation is of dimension four or greater, you can select any three-dimensional orthogonal projection to view.
- Remember that the computed points will be saved in memory, so be aware of number of points generated by looking at Time and Step Size.
- If your computer has a slow CPU or small memory, you can disable interactive 3-D rotations by unchecking `Real-Time Rotation & Animation` from `Real-Time` on the control tree of the Phase Portrait view.
- Notice that the mouse function is set to 3-D and radio buttons are added to the Tool Bar.
- After hitting Go, you can grab the image in the view with the mouse by holding down the left mouse button and spin the image around, or use the radio buttons for special rotations.
- You can stop rotations only from the radio button [] for stop on the Tool Bar.

In this lesson we will use the four-dimensional system of ODEs

$$\begin{aligned}\dot{x}_1 &= ax_3\\ \dot{x}_2 &= bx_4\\ \dot{x}_3 &= -ax_1\\ \dot{x}_4 &= -bx_2\end{aligned}$$

stored in the library of Phaser under the name `Harmonic Oscillators ODE` to illustrate several basic 3-D graphical utilities of Phaser. A typical orbit of this system winds around a two-dimensional torus in four-dimensional space. Three-dimensional orthographic projection (plotting the first three variables) of these orbits wind around cylinders. Below, we will draw two such orbits in 3-D and rotate them in real time.

»> `Equation -> ODE Library -> Harmonic Oscillators ODE:`

From `Equation` menu, select `ODE Library` and then click on `Harmonic Oscillators ODE`.

»> `Numerics:`

Press `Numerics` on the button bar to go to Numerics Editor.

»> `Phase Portrait:`

Click on `Phase Portrait` at the root of the control tree. Notice that currently the x1 variable is assigned to the X-Axis, x2 variable to the Y-Axis, and x3 to the Z-Axis. These choices are fine. In dimensions greater than three, Phaser provides the facility to project onto a three-dimensional subspace from higher dimensions. Any three of the variables xi can be assigned to the X-, Y-, and Z-axes for plotting.

»> `Window Size:`

From the tree, select `Window Size.`. Set X-Min = -3.5, X-Max = 3.5, Y-Min = -2.5, Y-Max = 2.5, and Z-Min = -3.0, Z-Max = 3.0. Apply.

»> `Axes & Grids:`

From the tree, select `Axes & Grids.` Uncheck the buttons to the far left of `Display Graph XY-Axes` and `Display Graph XY-Grid.` Apply. Now the axes and grids will not be displayed on the Phase Portrait view.

»> `Connect Points:`

From the tree, select `Graphics.` Check the small box to the left of `Connect Points.` Apply. Now, two consecutive computed points on an orbit will be connected.

»> `Initial Conditions:`

Click on `Initial Conditions.` The first set of initial conditions is fine. To add a second set, click on the left-most square next to `Set 2` and enter the values x1 = 0.5, x2 = 1.5, x3 = 0.5, x4 = 1.5. Apply.

»> `Time:`

Click on the `Time` button and set `Stop Plotting` to 70.0. Apply.

»> `Plotting Step Size:`

Click on the `Algorithms` button and notice that the current algorithm is Dormand-Prince 5(4). Set `Plotting Step Size` to 0.05. Apply.

»> `Go:`

Now, return to the Main Application window and press `Go.` You should see two orbits being plotted.

Using the current Step Size of 0.05, we have computed the solutions for 70 time units. Therefore, we have generated 1400 points per solution. Since `Real Time Rotation & Animation` is checked by default, these 2800 points have been saved. Now, we are ready to twirl them around.

»> `3-D:`

Note that the mouse is set to be used for interactive rotations as indicated by `3-D` in the center of the tool bar. Now, while the cursor is in the Phase Portrait view, the mouse will be used for rotating the contents of this view. Notice that the shape of the cursor is changed as a reminder of this new assignment. To rotate orbits, move the cursor into the view and grab the orbits by pressing and holding the left mouse button. While the mouse button is pressed, move the cursor around in the view to rotate the orbits in the desired direction.

»> `Radio Button >:`

Now, hit the Radio Button > on the tool bar. The image should rotate continuously by itself at the speed of the last mouse drag. Watch the show.

»> `Radio Button []:`

Hit the Radio Button [] on the tool bar. The image should stop rotating.

»> `Radio Button X:`

Hit the Radio Button `X` on the tool bar. This is to request Phaser to rotate the image, from its last stopped position, about the X-Axis.

»> `Radio Button >:`

Hit the Radio Button > to start rotating the image about the X-Axis.

»> `Radio Button []:`

Hit the Radio Button [] on the tool bar. The image should stop rotating.

We will continue manipulating these orbits in the next lesson. If you would like to interrupt this lesson now, you can save the current state of Phaser as follows:

»> `File -> Save Current Project:`

From `File` menu of the Main Application window, select `Save Current Project.` In the pop-up dialog box, type a file name (tutorial10.ppf) and hit `Save.`

Lesson 11. Poincare Map in 3-D

This lesson is a continuation of the previous one. Here, we will slice the two orbits on cylinders with a specified plane and compute the points of intersections with the plane.

The points of intersections of orbits of an ODE in three dimensions with a specified plane can be viewed as orbits of a map on the plane. This reduction of the dynamics of a three-dimensional ODE to a two-dimensional MAP, called the Poincare map, is an important technique for investigating the dynamics of an ODE in three dimensions.

If you have interrupted your lessons, you can first Load the Phaser project that you have saved at the end of the previous lesson as follows:

»> `File -> Open Phaser Project:`

From `File` menu of the Main Application window, select `Open Phaser Project.` In the pop-up dialog box, click on the saved file name (tutorial10.ppf) and hit `Open.`

»> `Poincare:`

Return to Numerics Editor and select Current View button. From the control tree of the `Phase Portrait` view, select `Poincare.`

»> `Display Orbit Cut-Plane and/or Poincare Map:`

Click on the small square to the left of this item. Notice `Orbit Cut-Plane Equation 1.0 x + 0.0 y + 0.0 z + 0.0 = 0.0`. The portion of an orbit on the positive side of this plane will be plotted in the color of its Initial Condition, while the portion on the negative side of the plane will be plotted in a darker shade of the same color. Cut-Plane Equation is in the unrotated coordinate system. Apply.

»> `Clear and Go:`

Two orbits in shades of blue and yellow should be visible.

»> `Display Poincare Map:`

Return to Numerics Editor and click on the small square to the left of this item. Now, in addition to the orbits, the points of intersections of the orbits with the specified plane will be plotted. Large red dots will mark the points when the orbits pierce the plane as they cross the plane from the negative to the positive side, and large yellow dots will mark the crossings from the positive to the negative side. The colors and the sizes of intersection points can be customized, as desired. Apply.

»> `Clear and Go:`

The two orbits and their points of intersections with the plane should be visible. Notice that the points are lined up along four lines.

»> `Plot Poincare Map Crossing Direction(s):`

From the mathematical viewpoint, it may be necessary to plot the intersection points as orbits cross the plane in one direction only. To accomplish such a task, click on the drop-down choice widget and select `Positive`. Now only the intersection points as the orbits cross the plane from the negative side to the positive will be plotted. Apply.

»> `Clear and Go:`

The two orbits and their positive crossings, the red dots, should be visible. Notice that the crossings are lined up along two lines.

»> `Orbit Cut-Plane Equation:`

Modify the text fields for the coefficients of Cut-Plane Equation to `0.0 x + 1.0 y + 0.0 z + 0.0 = 0.0.` Now, the orbits will be cut and the Poincare map will be computed using this new plane. Apply.

»> `Clear and Go:`

The two orbits and their positive crossings with the new plane, the red dots, should be visible. Notice that the crossings are lined up along two circles.

»> `Initial Conditions:`

If only the intersection points, Poincare map, is desired, we can set the colors of Initial Conditions to the background color. Set the colors of the Initial Conditions to black. Apply.

»> `Clear and Go:`

Now, only the positive crossing points along two circles should be visible.

Lesson 12. Poincaré Map of Forced Duffing ODE

In this lesson we will compute the Poincaré, or the period, map of the periodically forced Duffing's oscillator given by the equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = ax_1 - bx_2 - x_1^3 + c\cos(2\pi d\,t)$$

Periodically forced oscillators occupy a prominent place in applied dynamical dynamical systems. The technique we will employ in this lesson is applicable to other forced equations such as `Forced Vanderpol ODE` and `Forced Bridge ODE`.

The solutions of a periodically forced equation are invariant under translations in time by the period of the forcing term. Thus the qualitative dynamics of solutions can be studied by plotting the values of the solutions once every period. The resulting planar map is called the *Poincaré* or *the period, map*. If the Poincaré map of a solution approaches a fixed point, then the solution must be periodic with the same period as the period of the forcing term. If, however, the Poincaré map wanders in a bounded region without ever repeating itself then the solution must be non-periodic, or chaotic.

To compute the Poincaré map numerically, we take a step size an integer multiple of which is the period (in this example, the period of the forcing term is 1/d). Then simply plot the solution only at the integer multiples of the period.

»> `Equation -> ODE Library -> Forced Duffing ODE:`

From the Equation menu, select ODE Library and then pick `Forced Duffing ODE`.

»> `Equation -> View Current Equation:`

From the Equation Menu select the entry `View Current Equation` to see the text of the Forced Duffing ODE.

»> `View:`

The current view should be the Phase Portrait view.

»> `Parameters:`

Click on the `Parameters` button on the button bar of the Numerics Editor. Set the value of the parameters a = 1.0, b = 0.3 and c = 0.3, d = 0.2. Apply.

»> `Initial Conditions:`

Click on the `Initial Conditions` button on the button bar of the Numerics Editor. Notice that Set 1 is t = 0.0, x1 = 1.2 and x2 = 1.7, which is fine. To activate Set 2 click on the small button to the left of Set 2 and set t = 0.0, x1 = -1.2 and x2 = 1.7. Apply.

»> `Window Size:`

Click on the `Current View` button. From the tree of the Current View, which is Phase Portrait, select `Window Size`. By clicking on the text fields, set `X-Min` = -2.0, `X-Max` = 2.0, `Y-Min` = -1.5, and `Y-Max` = 1.5. Apply.

»> `Time:`

Click on the `Time` button. To discard the transient (early) part of the orbit, set `Start Plotting` to 66 and `Stop Plotting` to 333. Apply.

»> `Go:`

Bring the Phase Portrait view to the foreground by clicking on the title bar of the Main Application Window. Now hit `Go` to see two solutions, the first one in blue and the second in yellow. Notice that both solutions are periodic.

»> `Algorithms:`

Click on the `Algorithm` button on the button bar of Numerics Editor. Notice that the current algorithm is Dormand-Prince 5(4). The current `Plotting Step Size` is set to 0.01. The choices of the Algorithm is fine for the present calculations.

Now, we will prepare for the computation of the Poincaré map. Since the parameter `d` is set to 0.2, the period of the forcing term is 5.0. To get the Poincaré map, all we need to do is set `Plotting Step Size` to 5. Apply.

»> `Clear` and `Go:`

Bring the Phase Portrait view to the foreground and press `Clear` and `Go.` Notice that there are two blue dots on the right and two yellow dots on the left. Thus, both solutions are periodic with period twice the period of the forcing term.

»> `Graphics:`

To give these important dots a bit more prominence, return to the Numerics Editor and select `Current View.` From the control tree, select `Graphics.` The first entry on the right is `Graph Point Size.` Click on the drop down widget and select 2. Apply.

»> `Clear` and `Go:`

Now, you should see the same pairs of dots plotted in larger point size.

As the parameters are varied, the dynamics of `Forced Duffing ODE` can undergo remarkable changes. Indeed, for certain parameter values the system exhibits visible chaos.

»> `Parameters:`

Return to the Numerics Editor and change the value of the parameter b to 0.2. Apply. Note that the period of the forcing remains unchanged so we can use the current setup to compute the Poincaré map.

»> `Time:`

Click on the `Time` button and set `Stop Plotting` to 4444. Apply.

»> `Clear` and `Go:`

Now you should see many blue and yellow dots sprinkled around with no discernible pattern. This set is an example of a strange attractor.

»> `Plotting Step Size:`

To see the actual solutions, rather than the Poincaré map, click on `Algorithms` button on the Numerics Editor and set `Plotting Step Size` to 0.02. Apply.

»> `Clear` and `Go:`

These are non-periodic (chaotic) solutions of Forced Duffing ODE. You can hit `Stop` if you have a slow computer.

Lesson 13. Sequencing Slides

In this lesson, we will learn how to create a sequence of slides. We will use `Sierpinski-Knopp (IFS) MAP` to create nine slides picturing fractal curves with fractal dimesions ranging from 1 to 1.8 with increments of 0.1

»> `Equation -> MAP Library -> Sierpinski-Knopp (IFS) MAP:`

From `Equation` menu, select `MAP Library`. From the list, select `Sierpinski-Knopp (IFS) MAP`.

»> `Equation -> Load Equation Defaults:`

Again from `Equation` menu, select `Load Equation Defaults.` (You can also simply type `l` as a keyboard shortcut to load equation defaults.) Now, values of all settings will be loaded from a system file resulting in a nice fractal curve.

»> `Phaser -> Numerics -> Parameters:`

From `Phaser` menu of the Main Application window, select `Numerics -> Parameters` to bring up the panel of the Numerics Editor containing the parameter settings. Notice that the equation parameter is set to `d = 1.7097.` Parameter `d` is the dimension of the fractal curve.

»> `d = 1.0   [x] Sequence, using delta 0.1:`

Set `d = 1.0`, check the `Sequence` box, and type `0.1` for the increment value delta. Apply. Return to Phaser main application window.

»> `Phaser -> Gallery:`

From `Phaser` menu of the Main Application window, select `Gallery`. The co-primary window Gallery should pop. Gallery currently consists of four empty frames.

»> `Gallery -> Size -> 3x3:`

From `Gallery` menu of the Gallery co-primary window, set the gallery size to 3x3. Now, you should see 9 empty frames in the Gallery.

»> `Sequence:`

Click on the `Sequence` button on the Button Bar of the Gallery window. Now, each frame in the Gallery should be populated with fractal curves with increasing fractal dimension.

»> `Slideshow:`

Click on the `Slideshow` button on the Button Bar of the Gallery window. Now, a new window for playing slide shows should spawn. Click on the `Play` button to see the slide show. Adjust the speed as you desire, or play it in a loop.

»> `Save Current Gallery:`

You can save the current Gallery in a file using `Save Current Gallery` entry on the File menu of the Gallery co-primary window.

»> `Serialize Gallery Images to Disk:`

Slides can be saved to disk as a sequence of graphics files using `Serialize Gallery Images to Disk` on the File menu of the Gallery window. Once saved, these images can be further processed using, for example, QuickTime.