Let a be a positive parameter and consider the initial-value problem

$$\frac{dy}{dx} = \frac{-y}{\sqrt{a^2 - y^2}}, \qquad y(x_0) = y_0, \qquad 0 < y_0 < a. \tag{1}$$

By the existence and uniqueness theorem, we notice that

1. a solution $y(x)$ exists, and
2. the solution $y(x)$ must be decreasing and positive on its interval of definition and hence y is defined on $[x_0, +\infty)$.

The differential equation is separable, hence

$$\int_{y_0}^{y} \frac{\sqrt{a^2 - z^2}}{z}\, dz = -\int_{x_0}^{x} dt = x_0 - x. \tag{2}$$

To integrate the left-hand side, we make the trigonometric substitution

$$z = a \sin\theta \qquad \text{and} \qquad dz = a\cos\theta\, d\theta \qquad \text{for} \qquad 0 \le \theta \le \pi/2$$

to obtain

$$\begin{aligned}
\int_{\phi_0}^{\phi} \frac{\sqrt{a^2 - a^2\sin^2\theta}}{a\sin\theta} a\cos\theta\, d\theta &= \int_{\phi_0}^{\phi} \frac{a\cos^2\theta}{\sin\theta}\, d\theta \\
&= a\int_{\phi_0}^{\phi} \frac{1 - \sin^2\theta}{\sin\theta}\, d\theta \\
&= a\int_{\phi_0}^{\phi} (\csc\theta - \sin\theta)\, d\theta \\
&= a(\cos\theta - \ln(\csc\theta + \cot\theta))|_{\phi_0}^{\phi}
\end{aligned}$$

.

A straightforward trigonometric analysis shows that

$$\cos\theta = \frac{\sqrt{a^2 - z^2}}{a}, \qquad \cot\theta = \frac{\sqrt{a^2 - z^2}}{z}, \qquad \text{and} \qquad \csc\theta = \frac{a}{z}.$$

Hence the left-hand side of (2) reduces to

$$\sqrt{a^2 - z^2} + a\ln(z) - a\ln\left(a + \sqrt{a^2 - z^2}\right)\Big|_{y_0}^{y}.$$

And thus the solution of the initial-value problem (1) is given implicitly by

$$\sqrt{a^2 - y_0^2} - \sqrt{a^2 - y^2} - a\ln(y/y_0) + a\ln\left(\frac{a + \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y_0^2}}\right) = x - x_0.$$